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Unistar's

Fully Solved Previous Years' Papers

Mathematics

B.A./B.Sc-III

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by

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Preface

In *Unistar's Solved Previous Years' Papers Mathematics B.Sc-III,* I have tried to make the solutions simpler and easily understandable to our students for whom this book has been written. Chapters have been divided as per new semester system. My esteemed friends, while patronizing this book for their classroom instruction will certainly enjoy it. I am confident that students will earn a lot of benefit out of it and make this book as their permanent companion.

I am very thankful to Prof. Anil Makkar (D.A.V. College, Abohar), Dr. Neela Pawar (MCM D.A.V. College for Women, Chandigarh) and Dr. Neeraj Chamoli (D.A.V. College, Chandigarh) for their constructive guidance. I am also thankful to the editorial team of Unistar Books who efficiently worked in bringing up the book.

I have taken care to ensure that the text is free from errors. If a learner finds any error in the text, please send the same. I shall welcome your constructive criticism and suggestions to update the book from time to time.

Rakesh Kumar

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April 2015 Paper

B.A. /B.Sc. (General) 3rd Year
April 2015
MATHEMATICS
Paper—I: Analysis

SECTION–A

I. (a) Define equivalent set and prove that set of integers is equivalent to set of natural numbers.

Sol. A set A is equivalent to set B, written as A ~ B

If there exists a function $f : A \to B$ which is one-one and onto

Let $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ be set of integers then $\exists$ a function $f : N \to Z$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & ; \text{If n is even natural number} \\ -\frac{n-1}{2} & ; \text{If n is odd natural number} \end{cases}$$

Here f is one-one onto

$\therefore \quad Z \sim N$

(b) Evaluate $\underset{n\to\infty}{\text{Lt}}$ L (P,f), where $f(x) = x^2$ and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}\right\}$

Sol. $f(x) = x^2$ is monotonically increasing and bounded in [0, 1].

Divide [0, 1] into n equal parts, each of length $\frac{1}{n}$, so that

$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}$ is a partition of [0, 1]

Comparing it with $P = \{x_0 = 0, x_1, x_2 \ldots, x_n = 1\}$, we have,

$x_i = \frac{i}{n}$ for i = 1, 2, 3, …., n.

Now $\delta_i = \frac{1}{n}$

$m_i = f(x_{i-1}) = (x_{i-1})^2 = \left(\frac{i-1}{n}\right)^2$ and $M_i = f(x_i) = x_i^2 = \left(\frac{i}{n}\right)^2$

$$L(P, f) = \sum_{i=1}^{n} m_i \delta_i = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (i - 2i + 1) = \frac{1}{n^3}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + n\right]$$

$$= \frac{1}{n^3}[(1^2 + 2^2 + \ldots + n^2) - 2(1 + 2 + 3 + \ldots + n) + n]$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$$

$$= \frac{1}{n^2}\left[\frac{(n+1)(2n+1)}{6} - n - 1 + 1\right]$$

$$= \frac{1}{n^2}\left[\frac{(n+1)(2n+1) - 6n}{6}\right] = \frac{1}{n^2}\left[\frac{2n^2 + 3n + 1 - 6n}{6}\right]$$

$$= \frac{1}{n^2}\left[\frac{2n^2 - 3n + 1}{6}\right] = \frac{(n-1)(2n-1)}{6n^2}$$

$$\lim_{n\to\infty} L(P, \textstyle\int) = \lim_{n\to\infty} \frac{\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)}{6} = \frac{2}{6} = \frac{1}{3}$$

II. (a) Prove that $0 \le \int_0^{\pi} \{x^2 + (\sin x)^3\}\, dx \le \pi(\pi^2 + 1)$

Sol. Let $f(x) = x^2 + \sin^3 x$

Now $f(x)$ is strictly increasing function in $[0, \pi]$

$m = g.l.b\ \{f(x) : 0 \le x \le \pi\}$

$= g.l.b\ \{x^2 + \sin^3 x : 0 \le x \le \pi\}$

$= 0^2 + \sin^3 0 = 0 + 0 = 0$

$M = l.u.b\ \{f(x) : 0 \le x \le \pi\}$

$= l.u.b\ \{x^2 + \sin^3 x : 0 \le x \le \pi\}$

$= \pi^2 + \sin^2\pi = \pi^2 + 0 = \pi^2$

By first Mean Value Theorem.

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\therefore \quad 0(\pi - 0) \le \int_0^{\pi} (x^2 + \sin^3 x)\, dx \le \pi^2(\pi - 0)$$

$$\Rightarrow \quad 0 \le \int_2^{\pi} (x^2 + \sin^3 x)\, dx \le \pi^3 + \pi$$

$$\Rightarrow \quad 0 \le \int_0^{\pi} (x^2 + \sin^3 x)\, dx \le \pi(\pi^2 + 1)$$

(b) Show that $\int_0^{\infty} \left(\frac{1}{x} - \frac{1}{\sinh x}\right) \frac{dx}{x}$ **is convergent.**

Sol. $$f(x) = \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} = \frac{\sinh x - x}{x^2 \sinh x} = \frac{\left(x + \frac{x^3}{\lfloor 3} + \frac{x^3}{\lfloor 5} + \ldots\right) - x}{x^3 \frac{\sinh x}{x}}$$

$$\lim_{x\to\infty} f(x) = \frac{1}{6}$$

$\therefore$ 0 is not a point of infinite discontinuity of f

$$\therefore \int_0^{\infty}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x} = \int_0^{1}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x} + \int_0^{\infty}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x}$$

$= \text{proper integer} + I_2$

$$I_2 = \int_1^{\infty}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x} = \int_1^{\infty}\frac{\sinh x - x}{x^2 \sinh x}\,dx$$

$$f(x) = \frac{\sinh x - x}{x^2 \sinh x} = \frac{e^x - e^{-x} - 2x}{x^2 (e^x e^{-x})} = \frac{1}{x^2} - \frac{2e^x}{x(e^{2x} - 1)} = \frac{1}{x^2} - \frac{2e^x}{x(1 - e^{-2x})}$$

Take $\phi(x) = \frac{1}{x^2}$ i.e. $\frac{f(x)}{\phi(x)} = 1 - \frac{2xe^{-x}}{1 - e^{-2x}}$

$\Rightarrow$ a finite quantity as $x \to \infty$

But $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent i.e. I_2 converges at ∞.

III. (a) Prove that $\int_0^1 \frac{1}{\sqrt{1 - x^4}}\,dx = -\frac{1}{4\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2$

Sol. Let $\int_0^1 \frac{1}{\sqrt{1 - x^4}}\,dx$

Put $x^2 = \sin\theta$, $\therefore x = (\sin\theta)^{\frac{1}{2}}$ $\Rightarrow dx = \frac{1}{2}(\sin\theta)^{-\frac{1}{2}}\cos\theta\, d\theta$

Now $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}} \frac{1}{2} (\sin\theta)^{-\frac{1}{2}} \cos\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta} (\sin\theta)^{-\frac{1}{2}} \cos\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{1}{2}} (\cos\theta)^0 \, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2} \frac{\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)\Gamma\left(\frac{0-1}{2}\right)}{\Gamma\left(\frac{\frac{-1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\frac{\sqrt{\pi}\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\sqrt{2}\pi} \qquad \left[\because \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi\right]$$

$$= \frac{1}{8}\frac{\sqrt{2}}{\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

(b) Prove that $\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi(a^2+b^2)}{4a^3b^3}$

Sol. We have $\int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x} dx = \frac{\pi}{2ab}$ (1)

Diff. both sides w.r.t. 'a' and using Leibnitiz's rule

$$\int_0^{\pi/2} \frac{-2a\cos^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2} dx = -\frac{\pi}{2a^2b}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\cos^2 x}{\left(a^2\cos^2 x + b^2\cos^2 x\right)^2} dx = \frac{\pi}{4a^3b} \qquad (2)$$

Similarly diff. (1) w.r.t. 'b' using Leibnitz's rule

$$\int_0^{\pi/2} \frac{\sin^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2} dx = \frac{\pi}{4ab^3} \qquad (3)$$

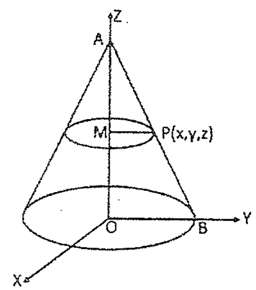
Adding (2) and (3)

$$\int_0^{\pi/2} \frac{\cos^2 x + \sin^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2} dx = \frac{\pi}{4a^3b} + \frac{\pi}{4ab^3}$$

$$\therefore \int_0^{\pi/2} \frac{1}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2} dx = \frac{\pi}{4ab}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \frac{\pi\left(a^2+b^2\right)}{4a^3b^3}$$

IV. (a) Find the volume of truncated cone with end radii "a" and "b" and height "h" (a > b).

Sol. Consider the cross-section of the cone by a variable plane perpendicular to z-axis and passing through the point P(x, y, z) on the cone. Obviously it is a circle. Let OM = z, where M is the centre of the circle which in the case is the above mentioned cross-section of the cone. Obviously $0 \le z \le h$. Radius of the circle of section = MP = R (say). Now from similar triangle AMP and AOB, we have



$$\frac{AM}{AO} = \frac{MP}{OB}$$

i.e. $\frac{h-z}{h} = \frac{R}{a}$

$[\because OA = h, OM = z,$

$\therefore AM - h - z$

and $OB = a, MP = R]$

$\therefore R = \frac{a}{h}(h-z)$

Since M lies on the z-axis where OM = z

$\therefore$ M = (0, 0, z) is the centre of the circle of section

$\therefore$ equation of the circle of section is

$$x^2 + y^2 = \left(\frac{a}{h}(h-z)\right)^2$$

Hence $V = \left\{(x,y,z); 0 \le z \le h, x^2+y^2 \le \frac{a^2}{h^2}(h-z)^2\right\}$,

$$= \left\{(x,y,z); 0 \le z \le h, -\frac{a}{h}(h-z) \le y \le \frac{a}{h}(h-z)\right.$$

$$-\sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2} \le x \le \sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}\Bigg\}$$

$\therefore$ Volume = $\iiint\limits_V 1.dx\,dy\,dz$

$$= \int_0^h \left(\int_{-\frac{a}{h}(h-z)}^{\frac{a}{h}(h-z)} \left(\int_{-\sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}}^{\sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}} 1.dx \right) dy \right) dz$$

$$= 2\int_0^h \left(\int_{-\frac{a}{h}(h-z)}^{\frac{a}{h}(h-z)} \sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}\,dy \right) dz$$

$$= 4\int_0^h \left(\int_0^{\frac{a}{h}(h-z)} \sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}\,dy \right) dz$$

$$= 4\int_0^h \frac{\pi}{4}\frac{a^2}{h^2}(h-z)^2\,dz$$

$$= \frac{\pi a^2}{h^2}\int_0^h (h-z)^2\,dz$$

$$= \frac{\pi a^2}{h^2}\left[\frac{(h-z)^3}{3(-1)}\right]_0^h$$

$$= \frac{1}{3}\pi a^2 h$$

$\Rightarrow$ volume of cone with radius r and height $h = \frac{1}{3}\pi r^2 h$

Given truncated cone with base radii 'a' and 'b' and height 'h'

$\therefore$ AD = a, BC = b, AB = h

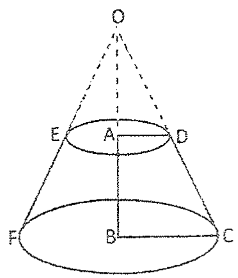
Δ's OAD and OBC are similar

$\Rightarrow \frac{OA}{AD} = \frac{OB}{BC}$

Let OA = α

$\Rightarrow \frac{\alpha}{a} = \frac{\alpha + h}{b}$

$\Rightarrow \alpha b = \alpha a + ah$



$\Rightarrow \alpha = \frac{ah}{b-a} \Rightarrow OA = \frac{ah}{b-a}$

Volume of truncated cone

= Volume of cone OFC – Volume of cone OED

$$= \frac{1}{3}\pi b^2 (OB) - \frac{1}{3}\pi a^2 (OA)$$

$$= \frac{1}{3}\pi\left[b^2\left(\frac{ah}{b-a} + h\right) - a^2\left(\frac{ah}{b-a}\right)\right]$$

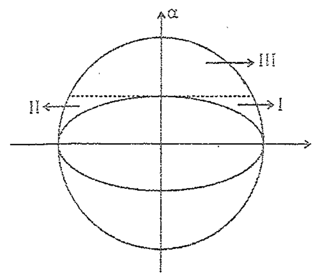
$$= \frac{1}{3}\pi\left[\frac{b^3 h - a^3 h}{b-a}\right]$$

$$= \frac{1}{3}\pi\frac{(b-a)(b^2+ab+a^2)}{b-a}h$$

$$= \frac{1}{3}\pi(b^2+ab+a^2)h$$

(b) **Change the order of integration** $\int_{-a}^{a}\int_{\frac{\sqrt{a^2-x^2}}{2}}^{\sqrt{a^2-x^2}} f(x,y)\,dydx$**. Hence evaluate it when $f(x, y) = 1$.**

Sol: Given Integral is $\int_{-a}^{a}\int_{\frac{\sqrt{a^2-x^2}}{2}}^{\sqrt{a^2-x^2}} f(x,y)dy\,dx$



We consider three regions I, II, III

In given integral y varies from $\frac{\sqrt{a^2-x^2}}{2}$ to $\sqrt{a^2-x^2}$

and then x varies from – a to a.

In change of order of integration

Region I : x varies from $\sqrt{a^2-4y^2}$ to $\sqrt{a^2-y^2}$

y varies from 0 to a/2

Region II : x varies from $-\sqrt{a^2-y^2}$ to $\sqrt{a^2-4y^2}$

y varies from 0 to a/2

Region III : x varies from $-\sqrt{a^2-y^2}$ to $\sqrt{a^2-y^2}$

y varies from a/2 to a

For f(x, y) = 1

$$\int_{-a}^{a}\int_{\sqrt{a^2-x^2}/2}^{\sqrt{a^2-x^2}} dy\,dx = \int_{0}^{a/2}\int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} dy\,dx + \int_{0}^{a/2}\int_{-\sqrt{a^2-4y^2}}^{-\sqrt{a^2-4y^2}} dx\,dy + \int_{a/2}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy\,dx$$

$$= \int_0^{a/2}\left(\sqrt{a^2-y^2}-\sqrt{a^2-4y^2}\right)dy\int_0^{a/2}\left(-\sqrt{a^2-4y^2}+\sqrt{a^2-y^2}\right)dy + \int_{a/2}^{a}\left(\sqrt{a^2-y^2}+\sqrt{a^2-y^2}\right)dy$$

$$= 2\int_0^{a/2}\sqrt{a^2-y^2}\,dy + 2\int_{a/2}^{a}\sqrt{a^2-y^2}\,dy - 2\int_0^{a/2}\sqrt{a^2-4y^2}\,dy$$

$$= 2\int_0^{a}\sqrt{a^2-y^2}\,dy - 2\int_0^{a/2}\sqrt{a^2-4y^2}\,dy$$

$$= 2\left[\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a - \frac{2}{2}\left[y\sqrt{a^2-4y^2}+\frac{a^2}{2}\sin^{-1}\frac{2y}{a}\right]_0^{a/2}$$

$$= 2\left[\frac{a^2}{2}\cdot\frac{\pi}{2}-0\right] - \left[\frac{a^2}{2}\cdot\frac{\pi}{2}-0\right] = \frac{a^2\pi}{4}$$

SECTION – B

V. (a) **Verify divergence theorem for $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2k$ taken over the region bounded by the cylinder $x^2 + y^2 = 4, z = 0, z = 3$**

Sol. Let S be the surface of the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$ and V be the volume enclosed by S.

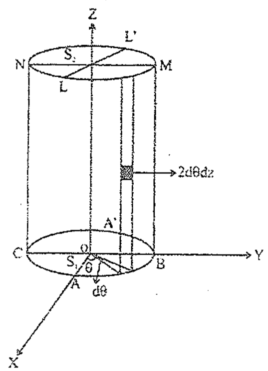
Now $$\iiint_V \nabla.\bar{A}\,dv = \iiint_V\left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}(-2y^2)+\frac{\partial}{\partial z}(-2z^2)\right]dV$$

$$= \iiint_V(4-4y+2z)dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z-4yz+\frac{2z^2}{2}\right]_{z=0}^{3}dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-12y+9)\,dy\,dx$$

$$= \int_{x=-2}^{2}\left[21y-\frac{12y^2}{2}\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$



$$= \int_{-2}^{2}\left[\left\{21\sqrt{4-x^2}-6(4-x^2)\right\}-\left\{21\sqrt{4-x^2}-6(4-x^2)\right\}\right]dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\int_0^2\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[\left(0+2.\frac{\pi}{2}\right)-(0+0)\right]$$

$= 84\pi$

The surface S of the cylinder consists of a basic $S_1(z = 0)$, the top $S_2(Z = 3)$ and the convex portion $S_3(x^2 + y^2 = 4)$, then

$$\iint_{S_1}\bar{A}.\bar{n}\,dS = \iint_{S_1}\bar{A}.\bar{n}\,dS_1 + \iint_{S_2}\bar{A}.\bar{n}\,dS_2 + \iint_{S_3}\bar{A}.\bar{n}\,dS_3 \quad ...(1)$$

On $S_1 : z = 0$, $\bar{n} = -\bar{k}$, $\bar{A} = 4x\hat{i} - 2y^2\hat{j}$

$\therefore$ $\bar{A}.\bar{n} = (4x\hat{i} - 2y^2\hat{j} + 0\hat{k}).(-\hat{k}) = 0$

$\therefore$ $\iint_{S_1}\bar{A}.\bar{n}\,dS = 0$

On $S_1 : z = 0$, $\bar{n} = -\bar{k}$, $\bar{A} = 4x\hat{i} - 2y^2\hat{j}$

$\therefore \bar{A}.\bar{n} = (4x\hat{i} - 2y^2\hat{j} + 0\hat{k}).(-\hat{k}) = 0$

$\therefore \iint_{S_1}\bar{A}.\bar{n} = dS_1 = 0$

On $S_2 : z = 3$, $\bar{n} = \bar{k}$, $\hat{k}$, $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

so that $\bar{A}.\bar{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}).\hat{k} = 9$

$\iint_{S_2}\bar{A}.\bar{n}\,dS_2 = \iint_{S_2}9\,dS_2 = 9\iint_{S_2}dS_2 = 9.(\pi.2^2)$ [$\because$ Area of circle of radius 2= $\pi 2^2$]

$= 36\pi$

On $S_3 = x^2 + y^2 = 4$ i.e. $\phi = x^2 + y^2 - 4$

$\therefore$ unit normal to $S_3 = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{\hat{i}\frac{\partial}{\partial x}(x^2+y^2-4)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2-4)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2-4)}{|\nabla\phi|}$$

$$= \frac{2x\hat{i}+2y\hat{j}+0\hat{k}}{\sqrt{4x^2+4y^2+0^2}} = \frac{2(x\hat{i}+y\hat{j})}{2.2} = \frac{x\hat{i}+y\hat{j}}{2} \qquad [\because x^2+y^2=4]$$

$$\therefore \bar{A}.\bar{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).\frac{x\hat{i}+y\hat{j}}{2} = 2x^2 - y^3$$

Putting $x = 2\cos\theta$, $y = 2\sin\theta$

From the figure $dS_3 = 2d\theta\, dz$

$$\therefore \iint_{S_3} \bar{A}.\bar{n}\, dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3]2dz\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{3} (16\cos^2\theta - 16\sin^3\theta)dz\, d\theta$$

$$= 16\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)[z]_{z=0}^{3}\, d\theta$$

$$= 48\int_{0=0}^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta$$

$$= 48\int_0^{2\pi}\cos^2\theta\, d\theta - 48\int_0^{2\pi}\sin^3\theta\, d\theta = 24\int_0^{2\pi}2\cos^2\theta\, d\theta - 48.0$$

$$\begin{bmatrix}\text{If} & f(\theta) = \sin^3\theta, \\ \text{then } f(2\pi-\theta) = [\sin(2\pi-\theta)]^3 = (-\sin\theta)^3 = -\sin^3\theta = -f(\theta)\end{bmatrix}$$

$$= 24\int_0^{2\pi}(1+\cos 2\theta)\, d\theta = 24\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 24\,[(2\pi+0)-(0+0)] = 48\pi$$

$$\therefore \text{from (1)}, \iint_S \bar{A}.\bar{n}\, dS = 0 + 36\pi + 48\pi = 84\pi$$

$$\therefore \iint_S \bar{A}.\bar{n}\, dS = \iiint_V \nabla\bar{A}\, dV$$

Hence divergence theorem is verified.

(b) **Define:**

(i) **Pointwise convergence of a sequence of a functions**

(ii) **Uniform convergence of a sequence of a functions**

Sol. (i) **Pointwise Convergence:** Suppose $< f_n >$, n = 1, 2, 3, ……… is a sequence of a function defined on [a, b], then the sequence $< f_n >$ converges point-wise to a real-valued function f defined on [a, b],l written as $f_n \xrightarrow{p.w.} f$ on [a, b]

If $\forall x \in [a,b], \lim_{n\to\infty} f_n(x) = f(x)$ i.e. $\forall x \in [a,b]$ and given any ε> 0 however small ∃ an integer N (dependent on x and ε) such that

$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \varepsilon$

(ii) **Uniform Convergence:** The sequence $< f_n >$, n = 1, 2, 3 ……., converge uniformly on [a, b] to a function f if for every ε> 0.

∃ an integer N (independent of x and depend on ε) such that

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \;\; \forall n \geq N$

It is clear from the definition that Uniform convergence ⇒ point-wise convergence and Uniform limit = point-wise limit.

(c) **Is pointwise convergence of a sequence of a functions uniformly convergent? Justify.**

Sol. Consider

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{nx}{1+n^2x^2} = \lim_{n\to\infty}\frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} = \frac{0}{0+x^2} = 0$$

$\Rightarrow f(x) = 0 \;\; \forall x \in \mathbb{R}$

Hence pointwise convergent

Now $|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \left|\frac{nx}{1+n^2x^2}\right|$

Let $y = \frac{nx}{1+n^2x^2}$

$$\frac{dy}{dx} = \frac{(1+n^2x^2)(n)-(nx)(2n^2x)}{(1+n^2x^2)^2} = \frac{n-n^3x^2}{(1+n^2x^2)^2}$$

For max or min $\frac{dy}{dx} = 0$

$\Rightarrow n - n^3x^2 = 0 \quad \Rightarrow x^2 = \frac{1}{n^2} \Rightarrow x = \frac{1}{n}$

Now $\frac{dy}{dx}=\frac{n-n^3x^2}{(1+n^2x^2)^2}$

$$\therefore \frac{d^2y}{dx^2}=\frac{(1+n^2x^2)^2(-2n^3x)-(n-n^3x^2)2(1+n^2x^2)(2n^2x)}{(1+n^2x^2)^4}=\frac{-2n^2x(3n^3x^2-n)}{(1+n^2x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n}}=\frac{-2n^2\left(\frac{1}{n}\right)\left(3n^3\frac{1}{n^2}-n\right)}{\left(1+n^2\frac{1}{n^2}\right)^3}=\frac{-2n(3n-n)}{(1+1)^3}=\frac{-4n^2}{8}=\frac{-n^2}{2}<0$$

$\Rightarrow$ y is maximum, when $x=\frac{1}{n}$

$$y_{max}=\frac{n.\frac{1}{n}}{1+n^2.\frac{1}{n^2}}=\frac{1}{1+1}=\frac{1}{2}$$

Thus $M_n=\sup_{x\in[a,b]}|f_n(x)-f(x)|=\sup_{x\in[a,b]}\left|\frac{nx}{1+n^2x^2}-0\right|=\sup_{x\in[a,b]}\left|\frac{nx}{1+n^2x^2}\right|=\frac{1}{2}$

Thus, M_n does not tend to zero as $n\to\infty$

$\therefore$ By M_n-test sequence $<f_n>$ is not uniformly convergent in any interval containing zero.

VI. **(a) Show that x = 0 is a point of non-uniform convergence of the series** $\sum_{n=1}^{\infty}\frac{x}{[(n-1)x+1][nx+1]}$.

Sol. The given series is

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots..$$

or $$\sum_{n=1}^{\infty}\frac{x}{[(n-1)x+1][nx+1]}$$

$$\therefore\quad u_n(x)=\frac{x}{[(n-1)x+1][nx+1]}=\frac{1}{(n-1)x+1}-\frac{1}{nx+1}$$

$$\therefore\quad u_1(x)=1-\frac{1}{x+1}$$

$$\therefore\quad u_2(x)=\frac{1}{x+1}-\frac{1}{2x+1}$$

$$\therefore\quad u_3(x)=\frac{1}{2x+1}-\frac{1}{3x+1}$$

....

$$\therefore\quad u_n(x)=\frac{1}{(n-1)x+1}-\frac{1}{nx+1}$$

Adding vertically, we get

$$f_n(x)=1-\frac{1}{nx+1}\to 1 \text{ as } n\to\infty \text{ and } x\neq 0$$

When $n\neq 0$:

$$M_n=\text{Sup}\{|f_n(x)-f(x)| : x \in R, x\neq 0\}$$
$$=\text{Sup}\left\{\left|1-\frac{1}{nx+1}-1\right| : x\in R,\ x\neq 0\right\}$$
$$=\text{Sup}\left\{\frac{1}{|nx+1|} : x\in R,\ x\neq 0\right\}$$
$$>\frac{1}{n.\frac{1}{n}+1}=\frac{1}{2}\neq 0$$

$\left[\text{Taking } x=\frac{1}{n}\right]$

Thus the sequence convergences non-uniformly.

Since $x\to 0$ as $n\to\infty$

$\therefore$ 0 is a point of non-uniform cgs.

i.e. the sequence does not converge uniforming in any interval including 0.

(b) State and prove Dirichlet's test for uniformly convergence of series of a functions.

Sol. $R_{n,p}(x)=u_{n+1}(x)\,v_{n+1}(x)+u_{n+2}(x)\,v_{n+2}(x)+\ldots..+u_{n+p}(x)\,v_{n+p}(x)$

$=[f_{n+1}(x)-f_n(x)]\,v_{n+1}(x)+[f_{n+2}(x)-f_{n+1}(x)]\,v_{n+2}(x)+\ldots..+[f_{n+p}(x)-f_{n+p-1}(x)]\,v_{n+p}(x)$

$=f_{n+1}(x)\,[v_{n+1}(x)-v_{n+2}(x)]+f_{n+2}(x)\,[v_{n+2}(x)-v_{n+3}(x)]+\ldots+f_{n+p-1}(x)[v_{n+p-1}(x)$
$-v_{n+p}(x)]+f_{n+p}(x)v_{n+p}(x)-f_n(x)v_{n+1}(x)$ ...(1)

Now all $v_1(x)$, $v_2(x)$, ... are + ve and $v_1(x)>v_2(x)>v_3(x)>\ldots\ldots$

and $|f_n(x)|<K$ for any x on [a, b] and for all n.

$(1)\Rightarrow |R_{n,p}(x)|\leq |\,|f_{n+1}(x)|\,[v_{n+1}(x)-v_{n+2}(x)]$
$+|f_{n+2}(x)|\,[v_{n+2}(x)+v_{n+3}(x)]+\ldots..+|f_{n+p-1}(x)|\,[v_{n+p-1}(x)-v_{n+p}(x)]+|f_{n+p}(x)|\,v_{n+p}(x)+|f_n(x)|\,v_{n+1}(x)$
$<K[v_{n+1}(x)-v_{n+p}(x)+v_{n+p}(x)+v_{n+1}(x)]$
$=2K\,v_{n+1}(x)$ ...(2)

Also since $\{v_n(x)\}$ cgs to zero.

$\therefore \quad v_{n+1}(x) < \dfrac{\varepsilon}{2K} \forall n \geq m$

Thus (2) $\Rightarrow |R_{n,p}(x)| < 2K\dfrac{\varepsilon}{2K} = \varepsilon$ for $n \geq m$

$\Rightarrow \quad |R_{n,p}(x)| < \varepsilon \ \forall \ n \geq m$ and for every x in [a, b]

Hence the series $\Sigma\, u_n(x)\, v_n(x)$ is uniformly cgs in [a, b].

VII. (a) Find the radius of convergence and interval of convergence of the power series $\sum_{n=2}^{\infty} \dfrac{(n-2)^{n-2}}{n \log n}$.

Sol. The given series is $\sum_{n=2}^{\infty} \dfrac{(n-2)^{n-2}}{n \log n}$

Put $x - 2 = y$

$\therefore$ the given power series is $\sum_{n=2}^{\infty} \dfrac{y^{n-2}}{n \log n}$

Put $n - 2 = m$ or $n = m + 2$

$\therefore$ the given power series is $\sum_{m=0}^{\infty} \dfrac{y^m}{(m+2)\log(m+2)}$

$$\therefore a_m = \frac{1}{(m+2)\log(m+2)},\quad a_{m+1} = \frac{1}{(m+3)\log(m+3)}$$

$$\therefore \left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(m+2)\log(m+2)}{(m+3)\log(m+3)}\right|$$

$$\therefore \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\frac{(m+2)\log(m+2)}{(m+3)\log(m+3)}$$

$$= \lim_{m\to\infty}\frac{m+2}{m+3}\lim_{m\to\infty}\frac{\log(m+2)}{\log(m+3)}$$

$$= \lim_{m\to\infty}\frac{1+\frac{2}{m}}{1+\frac{3}{m}}\lim_{m\to\infty}\frac{\frac{1}{m+2}}{\frac{1}{m+3}}$$

$$= \frac{1+0}{1+0}\lim_{m\to\infty}\frac{m+3}{m+2}$$

$$= 1 \times \lim_{m\to\infty}\frac{1+\frac{3}{m}}{1+\frac{2}{m}} = 1 \times \frac{1+0}{1+0} = 1 \times 1 = 1$$

Let R be radius of convergence

$$\therefore \frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| \Rightarrow \frac{1}{R} = 1 \Rightarrow R = 1$$

$\therefore$ Radius of convergence = 1

$\therefore$ interval of convergence $-1 < y < 1$

or $\quad -1 < x - 2 < 1$

or $\quad 1 < x < 3$

$\therefore$ interval of convergence (1, 3).

(b) Show that:

$$\frac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2} - \left(1+\frac{1}{2}\right)\frac{x^3}{3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4} + \ldots \quad -1 < x \leq 1.$$

Sol. We have

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n$$

Here $a_n = (-1)^n,\ a_{n+1} = (-1)^{n+1}$

$$\therefore \quad \underset{m\to\infty}{\text{Lt}}\left|\frac{a_n}{a_n+1}\right| = 1$$

$\therefore$ radius of convergence = 1 and interval of convergence is (–1, 1)

The series on the R.H.S is a power series which is absolutely convergent for $-1 < x < 1$. Hence it will be uniformly convergent in $(-\lambda, \lambda)$ for $|\lambda| < 1$. The power series does not converge for $x = \pm 1$.

The integral series will have the same characteristics.

Integrating both sides w.r.t x, we have.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots + c$$

For $\quad x = 0, c = 0$

$$\therefore \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

For x = 1, the power series on the R.H.S reduces to

$$-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots = \sum(-1)^{n-1}\left(\frac{1}{n}\right).$$

Which is an alternating series and hence convergent by Leibnitz's test. The above series does not converge for $x = -1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{.........} \quad \text{for } -1 < x \le 1 \qquad \text{...(2)}$$

(ii) For $x = 1$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{..........}$$

(iii) From (1) and (2) both the power series on the R.H.S converge absolutely in $(-1, 1)$. Therefore their Cauchy product will be also converge in $(-1, 1)$

$$\therefore \quad (1+x)^{-1}\log(1+x) = [1 - x + x^2 + x^3 + ...]\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{.........}\right]$$

$$= x - x^2\left(1 + \frac{1}{2}\right) - x^3\left(1 + \frac{1}{2} + \frac{1}{3}\right) + ... \text{ for } -1 < x \le 1$$ Integrating both sides w.r.t x from 0 to x, we have

$$\frac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2} - \frac{x^3}{3}\left(1 + \frac{1}{2}\right) + \frac{x^4}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + ... \text{ for } -1 < x \le 1$$

Because the power sides on the R.H.S. convergence for $x = 1$ by Leibnitz's test

$$\therefore \frac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2} - \frac{x^3}{3}\left(1 + \frac{1}{2}\right) + \frac{x^4}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + ... \text{ for } -1 < x \le 1$$

VIII. (a) Obtain the Fourier series in the interval $\left[\frac{-l}{2}, \frac{l}{2}\right]$ of the function f, defined as:

Sol. $$f(x) = \begin{cases} x - [x] - \frac{1}{2} \text{ if } x \notin Z \\ 0 \quad \text{otherwise} \end{cases}$$

$$f(x) = x - [x] - \frac{1}{2} \text{ when x is not an integer}$$

$$f(x) = k - [k] - \frac{1}{2} = k \notin I$$

$$f(k+1) = (k+1) - [k+1] - \frac{1}{2} = (k+1) - \{[k]+1\} - \frac{1}{2}$$

$$= (k+1) - [k] - 1 - \frac{1}{2} = k + [k] - ½ = f(k)$$

$\therefore$ f(x) is periodic with period = 1

$$f(-k) = -k - [-k] - \frac{1}{2} = -k - \{-[k] - 1\} - \frac{1}{2} = \frac{1}{2}$$

$$= -k + [k] + 1 - \frac{1}{2} = -f(k)$$

$\therefore$ f(x) is an odd function

$\therefore$ $a_0 = 0$ and $a_n = 0$

Now $0 \le x < \frac{1}{2} \Rightarrow [x] = 0$

$\therefore$ for $0 \le x < \frac{1}{2}$

$$f(x) = x - 0 - \frac{1}{2} = x - \frac{1}{2}$$

Here $l = \frac{1}{2}$

Now $$b_n = \frac{1}{l}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x).\sin\frac{n\pi}{l}x\,dx = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\sin(2n\pi x)\,dx$$

$$= 4\int_0^{\frac{1}{2}} f(x).\sin(2n\pi x)\,dx \qquad \{f(x).\sin 2n\pi x \text{ is an even function}\}$$

$$= 4\int_0^{\frac{1}{2}}\left(x - \frac{1}{2}\right)\sin(2n\pi x)\,dx$$

$$= 4\left[\left(x - \frac{1}{2}\right)\left(-\frac{\cos 2n\pi x}{2n\pi}\right)_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}}\frac{\cos 2n\pi x}{2n\pi}dx\right] = 4\left[0 - \frac{\cos\theta}{4n\pi}\right]$$

$$= -\frac{1}{n\pi}$$

Substituting the values, we have

$$f(x) = x - [x] - \frac{1}{2} = -\sum_{n=1}^{\infty}\left(\frac{\sin 2n\pi x}{n\pi}\right)$$

(b) Obtain the Fourier series for the function:

$$f(x) = \begin{cases} \pi x & \text{if } 0 \le x \le 1 \\ \pi(2 - x) & \text{if } 1 < x \le 2 \end{cases}$$

Sol. Here, $l = 1$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\left[\frac{1}{2}\right] + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_0^1 \pi x \cos n\pi x\,dx$$

$$= 2\int_0^1 \pi x \cos n\pi dx$$

$$= 2\left[\pi x.\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{\pi^2 n^2}\right)\right]_0^1$$

$$= 2\left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right)$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}\{(-1)^n - 1\}$$

$$= \begin{cases} \frac{-4}{n^2\pi}, & \text{n is odd} \\ 0, & \text{n is even} \end{cases}$$

In the second integral
Take $2 - x = t \Rightarrow dx = -dt$
$x = 2 \Rightarrow t = 0$
$x = 1 \Rightarrow t = 1$

$$\therefore \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \int_1^0 \pi(t)\cos(2n\pi - n\pi t)(-dt)$$

$$= \pi\int_0^1 t\cos n\pi t\,dt = \pi\int_0^1 x\cos n\pi x\,dx$$

(By the above substitution)

$$And \quad b_n = \int_0^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x \sin n\pi x\,dx - \int_0^1 \pi x \sin n\pi x\,dx = 0$$

$$= 0$$

$$\therefore \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \int_1^0 \pi t \sin(2n\pi - n\pi t)(-dt)$$

$$= -\int_0^1 \pi t \sin n\pi t\,dt$$

$$= -\int_0^1 \pi x \sin n\pi x\,dx$$

$$\therefore f(x) = \frac{1}{2}(\pi) + \sum_{n=0}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos n\pi x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

B.A. /B.Sc. (General) 3rd Year
April 2015
MATHEMATICS
Paper—II: Abstract Algebra

SECTION–A

1. (a) **Prove that set of fourth roots of unity forms an abelian group under multiplication of complex numbers.**

Sol. Given G = {1, –1, i, – i}

We make the following table first:

.	1	– 1	i	– i
1	1	– 1	i	– i
– 1	– 1	1	– i	1
i	i	– i	– 1	1
– i	– i	i	1	– 1

(i) **Closure Property.**
∀ a, b ∈ G ⇒ ab ∈ G [From table]

(ii) **Associative Law.**
∀ a, b, c ∈ G
(a, b). c = a. (b. c) [as (1. i) . (–i) = (i). (–i) = 1 and 1. (i. – i) = 1.1 = 1 and similarly others]

(iii) **Existence of Identity.**
∃ 1 ∈ G such that
a. 1 = a = 1. a ∀ a ∈ G. [as i. 1 = i = 1 .i]

(iv) **Existence of Inverse.**
∀ a ∈ G, ∃ b ∈ G such that
a. b = 1 = b. a [as (i) (–i) = 1 = (– i) (i)]

(v) **Commutative Law.**
∀ a, b ∈ G
a. b = b. a [as (–1) (i) = –i and i. (– i) = –i]

Hence < G, *> is an abelian group.

(b) Let N be a cyclic normal subgroup of a group G. Show that every subgroup of N is normal in G.

Sol. Let a be a generator of cyclic subgroup N of G.

Let H be a subgroup of N so that H is cyclic

[$\because$ Every subgroup of a cyclic group is cyclic]

Let a^n be a generator of H, where n is the least positive integer that $a^n \in H$

Let $h \in H$ and $g \in G$

Then $h = (a^n)^t$; where t is an integer

Now
$$ghg^{-1} = g(a^n)^t g^{-1}$$
$$= g(a^t)^n g^{-1}$$
$$= (ga^t g^{-1})(ga^t g^{-1}) \ldots\ldots \text{ n factors}$$
$$= (ga^t g^{-1})^n \qquad \ldots(1)$$

Also $a^t \in N$ [$\because$ a is a generator of N]

$\because$ N is normal in G

$\therefore$ $g \in G, a^t \in N \Rightarrow g a^t g^{-1} \in N$

Therefore $g a^t g^{-1}$ is expressible $ga^t g^{-1} = a^m$ for some +ve integer m.

(1) $\Rightarrow gh g^{-1} = (a^m)^n = (a^n)^m$

$\because$ a^n is a generator of H

$\therefore$ $g \in G, h \in H \Rightarrow gh g^{-1} \in H$

which proves that H is a normal in G.

II. (a) If H and K where $H \subseteq K$ are two subgroups of a finite group a then show that $[G : H] = [G : H] = [G : K]\,[K : H]$

Sol. If H is subgroup of a finite group G then index of H in G = the number of distinct right (or left) cosets of H in G

$$[G:H] = \frac{o(G)}{o(H)}$$

Since $H \subseteq K$ be two subgroups of a group G therefore H is also a subgroup of K

Also H is subgroup of finite group G.

Therefore

$$[G:H] = \frac{o(G)}{o(H)} = \frac{o(G)}{o(K)} \cdot \frac{o(K)}{o(H)}$$
$$= [G:K][K:H]$$

Hence $[G:H] = [G:K][K:H]$

(b) For any n > 1 the subset A_n of S_n of all even permutations is a normal subgroup of S_n of index 2.

Sol. Since identity permutation is even, A_n is a non empty subset of S_n.

Again, $f, g \in A_n \Rightarrow f, g$ are even permutations

$\Rightarrow f, g^{-1}$ are even permutations

$\Rightarrow fog^{-1}$ is even

$\Rightarrow fog^{-1} \in A_n$

or that A_n is a subgroup of S_n.

If $f \in A_n$ and $g \in S_n$ be any members then $g^{-1} o fog$ will be even permutation, showing that $g^{-1} o fog \in A_n$ or that A_n is a normal subgroup of S_n.

Let $G = \{1, -1\}$ be the group under multiplication.

Define a map $\varphi : S_n \to G$, s.t.,

$$\varphi(f) = \begin{cases} 1 & \text{if f is even permutation} \\ -1 & \text{if f is odd permutation} \end{cases}$$

then φ is an onto mapping as $S_n (n \geq 2)$ must contain even as well as odd permutations. (Identity permutation and (12) will be in S_n).

To show that φ is a homomorphism

Let $f, g \in S_n$ be any members.

Case (i): Both f, g are even, then fog is even

$$\varphi(fog) = 1 = 1.1 = \varphi(f)\varphi(g)$$

Case (ii): Both f, g are odd, then fog is even

$$\varphi(fog) = 1 = (-1)(-1) = \varphi(f)\varphi(g)$$

Case (iii): One of f, g is odd, other even.

Suppose f is odd and g is even, then fog is odd

$$\varphi(fog) = -1 = (-1)(1) = \varphi(f)\varphi(g)$$

hence φ is an onto homomorphism and thus by Fundamental theorem of homomorphism

$$G \cong \frac{S_n}{\text{Ker } \varphi}$$

Since $f \in \text{Ker } \varphi \Leftrightarrow \varphi(f) = 1$

$\Leftrightarrow$ f is even $\Leftrightarrow f \in A_n$

We have Ker $\varphi = A_n$

or that $G \cong \frac{S_n}{A_n}$

But $o(G) = 2 \Rightarrow o\left(\frac{S_n}{A_n}\right) = 2$

$$\Rightarrow \frac{o(S_n)}{o(A_n)} = 2$$

$\Rightarrow \dfrac{o(S_n)}{2} = o(A_n)$

Thus index of A_n in S_n is 2.

III. (a) If R is commutative ring with unity with characteristic 2 then show that $(a+b)^2 = a^2 + b^2 = (a-b)^2 \ \forall\ a, b \in R$.

Sol. R is commutative ring with characteristic 2

$\Rightarrow 2x = 0 \ \forall\ x \in R$

and $ab = ba \ \forall\ a, b \in R$

Now $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$

$= a^2 + 2ab + b^2$ $(\because ab = ba)$

$= a^2 + b^2$ $(\because 2ab = 0)$

Also $(a-b)^2 = (a-b)(a-b)$

$= a^2 - ab - ba + b^2$

$= a^2 - 2ab + b^2$ $(\because ab = ba)$

$= a^2 + b^2$ $(\because 2ab = 0)$

Hence $(a+b)^2 = a^2 + b^2 = (a-b)^2$

(b) If A, B and C are ideal of a ring R such that $B \subseteq A$ then show that:

$A \cap (B + C) = B + (A \cap C)$

Give an example to show that in general

$A \cap (B + C) \neq (A \cap B) + (A \cap C)$

Sol. Let $x \in A \cap (B + C)$ be any element

$\Rightarrow$ $x \in A$ and $x \in B + C$.

Let $x = b + c$ for some $b \in B$ and $c \in C$.

Since $B \subseteq A \Rightarrow b \in B \Rightarrow b \in A$

Now $x \in A$ and $b \in A \Rightarrow x - b \in A$ [$\because$ A is ideal]

$\in (b + c) - b \in A$ i.e., $c \in A$ but $c \in C \Rightarrow c \in A \cap C$.

Thus $x = b + c \in B + (A \cap C)$.

$\therefore$ $A \cap (B + C) \subseteq B + (A \cap C)$. ...(1)

Conversely, let $y \in B + (A \cap C)$ be any element, then $y = b + k$, where $b \in B$ and $k \in A \cap C$ i.e., $k \in A$ and $k \in C$.

Now $b \in B$ and $k \in C$ $\Rightarrow b + k \in B + C$.

$\Rightarrow y \in B + C$.

Since $b \in B$ and $B \subseteq A \Rightarrow b \in A$ also $k \in A$

$\Rightarrow b + k \in A$

$\Rightarrow y \in A$.

Thus $y \in A$ and $y \in B + C \Rightarrow y \in A \cap (B + C)$.

$\therefore$ $B + (A \cap C) \subseteq A \cap (B + C)$. ...(2)

From (1) and (2) we get

$A \cap (B + C) = B + (A \cap C)$.

Moreover, as $B \subseteq A \Rightarrow A \cap B = B$. ...(3)

$\therefore$ from (3), we get

$A \cap (B + C) = B + (A \cap C) = (A \cap B) + (A \cap C)$.

Hence the required result is proved.

Here $B + C = \{0, b, c, a\}$ $(\because b + c = a)$

$= R$.

$\therefore$ $A \cap (B + C) = A \cap R = A = \{0, a\}$.

But $A \cap B = \{0\}$ and $A \cap C = \{0\}$.

$\therefore$ $(A \cap B) + (A \cap C) = \{0\}$.

Hence $A \cap (B + C) \neq (A \cap B) + (A \cap B)$.

IV. (a) An ideal P of a commutative ring R is prime if and only R/P is an integral domain.

Sol. Let P be a prime ideal of R

Let $(a+P)(b+P) = 0 + P$

Then $ab + P = P$

$\Rightarrow ab \in P$

$\Rightarrow a \in P$ or $b \in P$

$\Rightarrow a + P = P$ or $b + P = P$

thus $\dfrac{R}{P}$ is an integral domain.

Conversely, let $\dfrac{R}{P}$ be an integral domain.

Let $ab \in P$ then $ab + P = P$

$\Rightarrow (a+P)(b+P) = P$

$\Rightarrow a + P = P$ or $b + P = P$ ($\dfrac{R}{P}$ is integral domain)

$\Rightarrow a \in P$ or $b \in P$

Hence the result.

(b) If R is a commutative ring with unity and $f(x), g(x) \in R(x)$ then:

$\deg(f(x)\,g(x)) \leq \deg f(x) + \deg g(x)$

The equality holds if R is an I.D.

Sol. $\forall\ f(x), g(x) \in R[x]$, let

$f(x) = a_0 + a_1x + a_2x^2 + ...+ a_mx^m$ and

$g(x) = b_0 + b_1x + b_2x^2 + ...+ b_nx^n$, where $a_m \neq 0$, $b_n \neq 0$.

$\therefore$ deg $f(x) = m$, deg $g(x) = n$.

Then $f = (a_0, a_1, a_2, ..., a_m)$, $g = (b_0, b_1, b_2, ..., b_n)$,

where $a_m \neq 0$ and $a_i = 0$, $\forall\, i > m$ and

$b_n \neq 0$ and $b_i = 0$, $\forall\, i > n$. Then

(i) $f(g) = (c_0, c_1, c_2, ..., c_s)$

where $c_k = \sum_{i+j=k} a_i b_j,\ k \geq 0$

Now $c_{m+n+t} = a_0\, b_{m+n+t} + a_1\, b_{m+n+t-1} + ... + a_m\, b_{n+t} + ... + a_{m+t}\, b_n + ... + a_{m+n+t}\, b_0$
$= 0, \forall\, t \geq 1$

$\Rightarrow$ deg $f\,g \leq m + n =$ deg f + deg g.

Further, if R is integral domain. Then

$c_{m+n} = a_0\, b_{m+n} + a_1\, b_{m+n-1} + ... + a_m\, b_n + a_{m+1}\, b_{n-1} + ... + a_0\, b_{m+n}$
$= a_m\, b_n \neq 0$.

$\therefore$ deg $f\,g = m + n =$ deg f + deg g.

SECTION – B

V. (a) **Union of two subspaces is a subspace if and only if they are comparable.**

Sol. Given: $U_1 \cup U_2$ is a sub-pace of V(F)

To prove: Either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

Let us suppose that neither $U_1 \subseteq U_2$ nor $U_2 \subseteq U_1$

Now U_1 is not a subset of U_2 **(1)**

$\Rightarrow \exists x \in U_1$ such that $x \notin U_2$

Also U_2 is not a subset of U_1

$\Rightarrow \exists y \in U_2$ such that $y \notin U_1$ **(2)**

Further $x \in U_1 \Rightarrow x \in U_1 \cup U_2$

And $y \in U_2 \Rightarrow y \in U_1 \cup U_2$ $\left[\begin{matrix}\because U_1 \subset U_1 \cup U_2 \\ U_2 \subset U_1 \cup U_2\end{matrix}\right]$

But $U_1 \cup U_2$ is a subpace of V(F) [Given]

$\therefore x + y \in U_1 \cup U_2$

$\Rightarrow x + y \in U_1$ or $x + y \in U_2$ **(3)**

If $x + y \in U_1$ and $x \in U_1$

Then $(x + y) - x \in U_1$ [$\because U$ is a subsapce]

$\Rightarrow y \in U_1$ which contradicts (2)

$\therefore x + y \notin U_1$ **(4)**

Also if $x + y \in U_2$ and $y \in U_2$

Then $(x + y) - y \in U_2$ [$\because U_2$ is a subspace]

$\Rightarrow x \in U_2$ which contradicts (1)

$\therefore x + y \notin U_2$ **(5)**

Now (4) and (5) contradict (3)

$\therefore$ Our supposition is wrong

Hence either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

Conservely. Given : Either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

To prove : $U_1 \cup U_2$ is a subspace of V(F)

Since $U_1 \subseteq U_2$

$\therefore U_1 \cup U_2 = U_2$

But U_2 is a subspace of V(F)

$\therefore U_1 \cup U$ is a space of V(F)

Also $\because U_2 \subseteq U_1$

$\therefore U_1 \cup U_2 = U_1$

But U_1 is subspace of V(F)

$\therefore U_1 \cup U_2$ is a subspace of V(F)

Thus either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

$\Rightarrow U_1 \cup U_2$ is a subspace of V(F)

Hence the proof.

(b) **Determine whether $r(x) = 1 - 4x + 6x^2$ is in linear speed of p(x) and q(x) or not where $p(x) = 1 - x + x^2$ and $q(x) = 2 + x - 3x^2$.**

Sol. Let $r(x) = \alpha_1\, p(x) + \alpha_2\, q(x)$ for same scalars α_1, α_2.

$1 - 4x + 6x^2 = \alpha_1(1 - x + x^2) + \alpha_2\,(2 + x - 3x^2)$

$1 - 4x + 6x^2 = (\alpha_1 + 2\alpha_2) + (-\alpha_1 + \alpha_2)\,x + (\alpha_1 - 3\alpha_2)x^2$

Comparing the coefficients to like powers of x, we get

$\alpha_1 + 2\alpha_2 = 1$ ......(A)

$-\alpha_1 + \alpha_2 = -4$ ......(B)

$\alpha_1 - 3\alpha_2 = 6$ ......(C)

From (A) and (B),

$\alpha_1 = 3$ and $\alpha_2 = -1$

which also satisfies the equation (c)

Hence $r(x) = 3\,p(x) - q(x)$

$\Rightarrow$ r(x) is in linear span of p(x) and r(x)

VI. (a) Find the basis and dimension of subspaces of R^4 (R) spann by vectors: $x_1 = (1, 3, -1, 2)$, $x_2 = (0, 11, -5, 3)$ $x_3 = (2, -5, 3,)$ and $x_4 = (4, 1, 1, 5)$ and extend it to a basis of R^4(R).

Sol. Consider the matrix A whose rows are the given vectors and reduce it to echelon form

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

Operate $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

Operate $R_3 \to R_3 + R_2$, $R_4 \to R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is an echelon form of A

Thus non zero rows of an echelon form of A, forms a basis of the zero space of A i.e. subspace of R^4.

Hence B = { (1, 3, –1, 2), (0, 11, –5, 3)} is basis and dimension is 2.

IInd Part

Since dim $R^4 = 4$

∴ We need four L.I vectors of R^4 over R which include the above vectors (1, 3, –1, 2), (0, 11, –5, 3)

Consider B_1 = { (1, 3, –1, 2), (0, 11, –5, 3), (0, 0, 1, 0), (0, 0, 0, 1)}

To show B_1 is L.I.

From the matrix, whose rows are the vectors of B_1

$$\text{i.e.} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is an echelon matrix having all four non zero rows.

∴ B_1 is L.I set.

Hence B_1 is a basis of R^4 which is extension of given basis.

(b) Let T : V → W be in a linear transformation and V be a find dimensional vector space over F. Then: Rank T + Nullity T = dim V

Sol. **Statement:** If T: V → W be a linear transformation from a finite dimensional vector space V(F) to W(F) then

dimV = Rank T + Nullity T

Let dim V = n

Since Null space of T is a sub space of V

∴ it is finite dimensional

Let $\dim(N(T)) = k \le n$

∴ Nullity T = k

⇒ Basis set of N(T) contains k elements

Let $B_1 = \{v_1, v_2, ..., v_k\}$ be the basis set of N(T).

By definition of N(T), $T(v_1) = 0$, $T(v_2) = 0, ...T(v_k) = 0$

Now, the set B_1 can be extended so that the extended set

$B_2 = \{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$ consisting of n elements, is a basis set of V.

We consider a set B_3,

$B_3 = \{T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$

The set B_3 consists of T images of (n – k) vectors.

If we prove that B_3 is a basis for R(T), the theorem will be proved.

We have to show that:

(i) B_3 is L.I. set **(ii) B_3 spans R(T)** **(iii) B_3 is L.I.**

Let $\alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n \in F$ such that

$\alpha_{k+1}T(v_{k+1}) + \alpha_{k+2}T(v_{k+2}) + ... + \alpha_n T(v_n) = 0$

$\Rightarrow T(\alpha_{k+1}v_{k+1} + \alpha_{k+2}v_{k+2} + ... + a_n v_n) = 0$ [∵ T is a L.T.]

$\Rightarrow \alpha_{k+1}v_{k+1} + \alpha_{k+2}v_{k+2} + ... + \alpha_n v_n \in N(T)$

Now since B_1 is a basis of N(T)

∴ every element of N(T) can be written as the linear combination of elements of B_1

$\Rightarrow \alpha_{k+1}v_{k+1} + \alpha_{k+2}v_{k+2} + ... + \alpha_n v_n = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k + (-\alpha_{k+1})v_{k+1} + (-\alpha_{k+2})v_{k+2} ..... + (-\alpha_n)v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_k = \alpha_{k+1} = \alpha_{k+2} = ... = \alpha_n = 0$ [∵ B_2 is a L.I. set]

∴ $\alpha_{k+1}T(v_{k+1}) + \alpha_{k+2}T(v_{k+2}) + ... + \alpha_n T(v_n) = 0$

$\Rightarrow \alpha_{k+1} = \alpha_{k+2} = ... = \alpha_n = 0$

⇒ B_3 is a L.I. set.

(ii) B_3 spans R(T)

Let y be any element of R(T)

$\therefore \exists\, x \in V$ such that $T(x) = y$

Also $\because x \in V$ and B_2 is the basis set of V

$\therefore\ x = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_k v_k + \beta_{k+1} v_{k+1} + \ldots + \beta_n v_n$

$\Rightarrow\ y = T(x) = T(\beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_k v_k + \beta_{k+1} v_{k+1} + \ldots + \beta_n v_n)$

$= \beta_1 T(v_1) + \beta_2 T(v_2) + \ldots + \beta_k T(v_k) + \beta_{k+1} T(v_{k+1}) + \ldots + \beta_n T(v_n)$

$= \beta_1 .0 + \beta_2 .0 + \ldots + \beta_k .0 + \beta_{k+1} T(v_{k+1}) + \ldots + \beta_n T(v_n)$

$= \beta_{k+1} T(v_{k+1}) + \ldots + \beta_n T(v_n)$

$\Rightarrow y \in R(T)$ is a linear combination of elements of B_3

$\therefore B_3$ spans R(T)

Hence B_3 is a basis of R(T)

$\Rightarrow$ dim R(T)) = n – k

= dim V – dim N(T)

$\Rightarrow$ Rank T + Nullity T = dim V

VII. (a) Let T be a linear operator on R^2

T(x, y) = (4x – 2y, 2x + y)

Find the matrix of T relative to basis B = {(1, 1), (–1, 0)} Also verify that:

[T; B] [V; B] = [T(V) ; B] for any vector V $\in R^2$

Sol. Firstly, we shall express any element

$v = (\alpha, \beta) \in R^2$ as a linear combination of the element of basis B.

Let $(\alpha, \beta) = a(1,1) + b(-1,0)$ for reals a and b

$\Rightarrow (\alpha, \beta) = (a - b, a)$

$\therefore\ \alpha = a - b,\ \beta = a$

$\Rightarrow a = \beta$ and $b = \beta - \alpha$

$\therefore\ (\alpha, \beta) = \beta(1,1) + (\beta - \alpha)(-1,0)$ **(1)**

Given $T : R^2 \to R^2$ defined as

$T(x, y) = (4x - 2y, 2x + y)$

and $B = \{(1,1), (-1,0)\}$ is a basis of R^2

Now $T(1,1) = (4-2, 2+1) = (2,3) = 3(1,1) + (3-2)(-1,0)$ [Using (1)]

$= 3(1,1) + 1(-1,0)$

and $T(-1,0) = (-4-0, 2(-1)+0)$

$= (-4,-2) = (-2)(1,1) + (-2+4)(-1,0)$ [Using (1)]

$= -2(1,1) + 2(-1,0)$

$\therefore\ [T : B] = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}^t = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

Which is the matrix of T relative to the basis B.

To verify $[T; B][v; B] = [T(v); B]$

Let $v = (x, y) \in R^2$

Then $v = (x, y) = y(1,1) + (y - x)(-1,0)$ [Using (1)]

$\therefore\ [v; B] = [y \quad y - x]^t = \begin{bmatrix} y \\ y - x \end{bmatrix}$

Now $T(v) = T(x, y)$

$= (4x - 2y, 2x + y)$ (by def. of T)

$= (2x + y)(1,1) + (2x + y - 4x + 2y)(-1,0)$ [Using (1)]

$= (2x + y)(1,1) + (-2x + 3y)(-1,0)$

$\therefore\ [T(v); B] = [2x + y \quad -2x + 3y]^t = \begin{bmatrix} 2x + y \\ -2x + 3y \end{bmatrix}$

L.H.S. = [T ; B] [v ; B]

$= \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y - x \end{bmatrix}$

$= \begin{bmatrix} 3y - 2(y - x) \\ y + 2(y - x) \end{bmatrix}$

$= \begin{bmatrix} 2x + y \\ -2x + 3y \end{bmatrix} = [T(v); B] = R.H.S.$

Hence the result is verified

(b) **Show that A = $\begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are similar matrices over C.**

Sol. The eigen values of A are given by $|B - \alpha I| = 0$

i.e. $\begin{vmatrix} -1-\alpha & 3 & 0 \\ 0 & 2-\alpha & 0 \\ 2 & 1 & -1-\alpha \end{vmatrix} = 0$

$\Rightarrow (1 + \alpha)^2 (2 - \alpha) = 0 \Rightarrow \alpha = -1, -1, 2$ are the eigen values of A.

Similarly eigen values of B are given by $|A - \alpha I| = 0$

i.e. $\begin{vmatrix} -1-\alpha & 2 & 0 \\ 0 & -1-\alpha & 0 \\ 0 & 0 & 2-\alpha \end{vmatrix} = 0$

$\Rightarrow (1+\alpha)^2 (2-\alpha) = 0 \Rightarrow \alpha = -1, -1, 2$ are the eigen values of B.

Since the matrices A and B have same eigen values they are similar.

VIII. (a) Prove that eigen vectors corresponding to distinct eigen values of a Hermitian matrix are orthogonal.

Sol. Let X_1, X_2 be two characteristic vectors corresponding to two distinct characteristic roots λ_1, λ_2 of a Hermitian matrix A.

Then $AX_1 = \lambda_1 X_1$ (1)

and $AX_2 = \lambda_2 X_2$ (2)

$\because$ A is Hermitian $\therefore$ λ_1, λ_2 are real.

Now $\lambda_1 X_2^\theta X_1 = X_2^\theta (\lambda_1 X_1)$

$= X_2^\theta (AX_1)$ [by (1)]

$= (X_2^\theta A^\theta) X_1$ [$\because A^\theta = A$]

$= (AX_2)^\theta X_1$

$= (\lambda_2 X_2)^\theta X_1$ [by (2)]

$= \bar{\lambda}_2 X_2^\theta X_1$

$= \lambda_2 X_2^\theta X_1$ [$\because \lambda_2$ is real]

$\therefore$ $\lambda_1 X_2^\theta X_1 = \lambda_2 X_2^\theta X_1$

$\Rightarrow (\lambda_1 - \lambda_2) X_2^\theta X_1 = O$

But $\lambda_1 - \lambda_2 \neq 0$

$\therefore$ $X_2^\theta X_1 = O$

$\Rightarrow X_1$ and X_2 are orthogonal.

(b) Let $T : R^4 \to R^4$ be defined as:
T(x, y, z, t) = (2x + y, 2y, 2z, 5t)
Find the characteristic polynomial and minimal polynomial of T.

Sol. Consider the usual basis

B = { (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)} of R^4

First find [T : B] determined by

T (x, y, z, t) = (2x + y, 2y, 2z, 5t)

T (1, 0, 0, 0) = (2, 0, 0, 0)
T (0, 1, 0, 0) = (1, 2, 0, 0)
T (0, 0, 1, 0) = (0, 0, 2, 0)
T (0, 0, 0, 1) = (0, 0, 0, 5)

$\Rightarrow [T : B] = A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Characteristic polynomial of A is

$|\alpha I - A| = \begin{bmatrix} \alpha-2 & -1 & 0 & 0 \\ 0 & \alpha-2 & 0 & 0 \\ 0 & 0 & \alpha-2 & 0 \\ 0 & 0 & 0 & \alpha-5 \end{bmatrix} = (\alpha-2)^3 (\alpha-5)$

We know that characteristic polynomial and minimal polynomial have same irreducible factors.

$\therefore$ m(α) must be one of the following polynomial

$m_1(\alpha) = (\alpha-2)(\alpha-5)$

$m_2(\alpha) = (\alpha-2)^2 (\alpha-5)$

$m_3(\alpha) = (\alpha-2)^3 (\alpha-5)$

Now $m_1(A) = (A-2I)(A-5I)$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0.$

$m_2(A) = (A-2I)^2 (A-5I) = (A-2I)[(A-2I)(A-5I)]$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0.$

$\therefore$ $m_2(A) = 0$

Thus $m(\alpha) = (\alpha-2)^2 (\alpha-5)$ is minimal polynomial of A.

But minimal polynomial of A and T are same.

$\therefore$ Minimal polynomial of T is

$m(\alpha) = (\alpha-2)^2 (\alpha-5)$

B.A. /B.Sc. (General) 3rd Year
April 2015
MATHEMATICS
Paper—III: Opt. (ii) Probability, Theory

SECTION–A

I. (a) **A girl throws a die. If she gets 5 to 6, she tosses a coin three times and notes the number of head. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with die?**

Sol. Let A be the event of getting 5 or 6 and B the event of getting 1, 2, 3 or 4 when a die is thrown.

$$P(A)=\frac{2}{6}=\frac{1}{3} \text{ and } P(B)=\frac{4}{6}=\frac{2}{3}$$

Let H represents the event of getting a head when a coin is tossed.

When a coin is tossed thrice the conditional probability of getting a head

$$P(H\mid A)={}^{n}C_r p^r.q^{n-r}={}^{3}C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2=\frac{3}{8}$$

When a coin is tossed once the conditional probability of getting a head

$$=P(H\mid B)=\frac{1}{2}$$

$\therefore$ total probability of getting a head is

$$P(H)=P(A).P(H\mid A)+P(B).P(H\mid B)$$

$$=\frac{1}{3}.\frac{3}{8}+\frac{2}{3}.\frac{1}{2}=\frac{1}{8}+\frac{1}{3}=\frac{11}{24}$$

$\therefore$ the probability that a head comes when 1, 2, 3 or 4 come on the die.

$$=\frac{P(B).P(H\mid B)}{P(H)}=\frac{\frac{2}{3}.\frac{1}{2}}{\frac{11}{24}}=\frac{1}{3}\times\frac{24}{11}=\frac{8}{11}\ Ans.$$

(b) **For any n events $A_1, A_2, \ldots\ldots, A_n$, prove that:**

$$P\left(\bigcap_{i=1}^{n} A_i\right)\geq\sum_{i=1}^{n}P(A_i)-(n-1).$$

Sol. (i) For any two events A_1, A_2

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$\Rightarrow P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$ (1)

By axiom (1) of probability

$P(A) \le 1 \quad \Rightarrow \quad -P(A) \ge -1$

and in particular $-P(A_1 \cup A_2) \ge -1$

$\Rightarrow -[P(A_1) + P(A_2) - P(A_1 \cap A_2)] \ge -1$

$\Rightarrow P(A_1 \cap A_2) \ge P(A_1) + P(A_2) - 1$ (2)

$\Rightarrow$ The result is true for $n=2$

Let us suppose that the result is true for $n=r$

i.e. $P\left(\bigcap_{i=1}^{r} A_i\right) \ge \sum_{i=1}^{r} P(A_i) - (r-1)$

Now consider n = r + 1

$P\left(\bigcap_{i=1}^{r+1} A_i\right) = P\left(\left(\bigcap_{i=1}^{r} A_i\right) \cap A_{r+1}\right) \ge P\left(\bigcap_{i=1}^{r} A_i\right) + P(A_{r+1}) - 1$ [Using (2)]

$\ge \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1$ [Using (3)]

$= \sum_{i=1}^{r} P(A_i) - (r+1-1)$

i.e. $P\left(\bigcap_{i=1}^{r+1} E_i\right) \ge \sum_{i=1}^{r+1} P(A_i) - r$

$\Rightarrow$ the result is true for $n=r+1$

Thus by mathematical induction the result is true $\forall n \in N$.

II. (a) The odds in favour of standing first of three students appearing at an examination are 1 : 2, 2 : 5 and 1 : 7 respectively. Find the probability that either of them stands first.

Sol. Let A, B and C denote event of 3 students standing First in an examination respectively.

Given $\dfrac{P(A)}{P(\bar{A})} = \dfrac{1}{2}, \ \dfrac{P(B)}{P(\bar{B})} = \dfrac{2}{5}, \ \dfrac{P(C)}{P(\bar{C})} = \dfrac{1}{7}$

$\Rightarrow \quad P(A) = \dfrac{1}{3}, \ P(B) = \dfrac{2}{7}, \ P(C) = \dfrac{1}{8}$

Required Probability $P(A \cup B \cup C) = 1 - P\left(\overline{A \cup B \cup C}\right)$

$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= 1 - \left(1-\dfrac{1}{3}\right)\left(1-\dfrac{2}{7}\right)\left(1-\dfrac{1}{8}\right)$

$= 1 - \left(\dfrac{2}{3}\right)\left(\dfrac{5}{7}\right)\left(\dfrac{7}{8}\right)$

$= 1 - \dfrac{5}{12} = \dfrac{7}{12}$

(b) Let f(x) be p.d.f. of a discrete random variable X, which assumes values.

Incomplete Statement

III. (a) Let the random variable X takes values $x_i = \dfrac{(-1)^i 2^i}{i}$ $P(X = x_i) = 2^{-i}$; i = 1, 2, 3, ...

Check if E(X) exists or not.

Sol. $E(X) = \sum_{i=1}^{\infty} x_i P(X = x_i) \sum_{i=1}^{\infty} \dfrac{(-1)^i 2^i}{i} \times 2^{-i}$

$= \sum_{i=1}^{\infty} \dfrac{(-1)^i}{i}$

$= -1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{5} + \ldots$

$= -\left[1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} \ldots\right]$

$= -\log 2$

$\therefore E(X) = -\log 2$

But $\sum_{i=1}^{\infty} |x_i| P(X = x_i) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots$ is not convergent.

$\therefore$ E(X) does not exists.

(b) Find relation between cumulants and moments.

Sol. Cumulant generating function k(t) is defined as

$k_x(t) = k_1 t + k_2 \dfrac{t^2}{2!} + k_3 \dfrac{t^3}{3!} + \ldots\ldots + k_r \dfrac{t^r}{r!} + \ldots\ldots = \log M_x(t)$

$= \log\left[1 + \mu_1' t + \mu_2' \dfrac{t^2}{2!} + \mu_3' \dfrac{t^3}{3!} + \ldots\ldots + \mu_r' \dfrac{t^r}{r!} + \ldots\ldots\right]$

$= \log\left[\left(\mu_1' t + \mu_2' \dfrac{t^2}{2!} + \mu_3' \dfrac{t^3}{3!} + \mu_4' \dfrac{t^4}{4!} + \ldots\ldots\right)\right]$

$-\dfrac{1}{2}\left(\mu_1' t + \mu_2' \dfrac{t^2}{2!} + \mu_3' \dfrac{t^3}{3!} + \mu_4' \dfrac{t^4}{4!} + \ldots\ldots\right)^2$

$+\dfrac{1}{3}\left(\mu_1' t + \mu_2' \dfrac{t^2}{2!} + \mu_3' \dfrac{t^3}{3!} + \mu_4' \dfrac{t^4}{4!} + \ldots\ldots\right)^3$

$$-\frac{1}{4}\left[\left(\mu_1't+\mu_2'\frac{t^2}{2!}+\mu_3'\frac{t^3}{3!}+\mu_4'\frac{t^4}{4!}+\dots\dots\right)^4+\dots\dots\right]$$

Comparing the coefficients of like powers of t on both sides, we get

$k_1 = \mu_1' = \text{Mean}$

$$\frac{k_2}{2!}=\frac{\mu_2'}{2!}-\frac{\mu_1'^2}{2!} \Rightarrow k_2=\mu_2'-\mu_1'^2=\mu_2=\text{Varience}$$

$$\frac{k_3}{3!}=\frac{\mu_3'}{3!}-\frac{1}{2}\frac{2\mu_1'\mu_2'}{2!}+\frac{\mu_1'^3}{3} \Rightarrow k_3=\mu_3'-3\mu_2'\mu_1'+2M_1'^3=\mu_3$$

IV. (a) Find the recurrence relation $\mu_{r+1}=pq\left(nr\mu_{r-1}\frac{d\mu_r}{dp}\right)$ for the moments of binomial distribution.

Sol. By def. $\mu_r = E\{X-E(X)\}^r = \sum_{x=0}^{\infty}(x-np)^r P(X=x)$

$$=\sum_{x=0}^{\infty}(x-np)^r\, {}^nC_x p^x q^{n-x}$$

Differentiating with respect to p, we get

$$\frac{d\mu_r}{dp}=\sum_{x=0}^{n}{}^nC_x[-nr(x-np)^{r-1}p^x q^{n-x}+(x-np)^r\{xp^{x-1}q^{n-x}-(n-x)p^x q^{n-x-1}\}]$$

$$=-nr\sum_{x=0}^{n}{}^nC_x(x-np)^{r-1}p^x q^{n-x}\sum_{x=0}^{n}{}^nC_x(x-np)^r p^x q^{n-x}\left(\frac{x}{p}-\frac{n-x}{q}\right)$$

$$=-nr\sum_{x=0}^{n}(x-np)^{r-1}p(x)+\sum_{x=0}^{n}(x-np)^r p^x\left(\frac{x-np}{pq}\right)$$

$$\left[\because p(x)={}^nC_x p^x q^{n-x} \text{ and } p+q=1\right]$$

$$=-nr\sum_{x=0}^{n}(x-np)^{r-1}p(x)+\frac{1}{pq}\sum_{x=0}^{n}(x-np)^{r+1}p^x$$

$$=-nr\mu_{r-1}+\frac{1}{pq}\mu_{r+1}$$

$$\Rightarrow \mu_{r+1}=pq\left(nr\,\mu_{r-1}+\frac{d\mu_r}{dp}\right)$$

Which is the **Renovsky Formula**.

(b) In a lengthy manuscript, it is discovered that only 13.5% of the pages contain no typing errors. If we assume the number of errors per page to be a random variable with Poisson distribution, find the percentage of pages that have exactly one error.

Sol. Let X denote the number of errors on a randomly selected page then

$X \sim P(\alpha)$

Given that $P(X=0)=e^{-\alpha}=.135$

$\Rightarrow \quad \alpha \approx 2$ (approx)

Hence $P(X=1)=e^{-\alpha}\frac{\alpha}{1!}\approx .135\times 2=.270$

That is, we expect 27% of the pages to contain exactly one error.

SECTION-B

V. (a) Prove that mean deviation about mean of an exponential distribution is $\frac{2}{\lambda}e^{-1}$.

Sol. M.D. about mean $=E[(X-E(X))]=\int_0^{\infty}\left|x-\frac{1}{\lambda}\right|f(x)\,dx.$

$$=\lambda\int_0^{\infty}\left|x-\frac{1}{\lambda}\right|e^{-\lambda x}dx=\int_0^{\infty}|\lambda x-1|e^{-\lambda x}dx \quad \text{Put } \lambda x=y\Rightarrow dx-\frac{dy}{\lambda}$$

$$=\frac{1}{\lambda}\int_0^{\infty}|y-1|e^{-y}dy=\frac{1}{\lambda}\left[\int_0^1(1-y)e^{-y}dy+\int_1^{\infty}(y-1)e^{-y}dy\right]$$

$$=\frac{1}{\lambda}[e^{-1}+e^{-1}]=\frac{2}{\lambda}e^{-1}$$

(b) Show that mode of beta distribution is $\frac{a-1}{a+b-2}$

Sol. We know that, mode of a distribution is solution of $\frac{df}{dx}=0$ and $\frac{d^2f}{dx^2}<0$

Here, $f(x)=\begin{cases}\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, & 0<x<1\\ 0 & \text{, elsewhere}\end{cases}$

$$\therefore \frac{df}{dx}=\frac{1}{B(a,b)}[(a-1)x^{a-2}(1-x)^{b-1}+(b-1)x^{a-1}(1-x)^{b-2}(-1)]$$

$$=\frac{1}{B(a,b)}x^{a-2}(1-x)^{b-2}[(a-1)+(1-a-b+1)x]$$

$$=\frac{1}{B(a,b)}x^{a-2}(1-x)^{b-2}[(a-1)+(2-a-b)x]$$

and $$\frac{d^2f}{dx^2}=\frac{1}{B(a,b)}[(a-2)x^{a-3}(1-x)^{b-2}][(a-1)+(2-a-b)x]$$

$$+(b-2)x^{a-2}(1-x)^{b-3}(-1)+x^{a-2}(1-x)^{b-2}[(2-a-b)]$$

$$=\frac{1}{B(a,b)}[(a-2)x^{a-3}(1-x)^{b-2}[(a-1)+(2-a-b)x]$$

$$+x^{a-2}(1-x)^{b-3}+[(1-x)(2-a-b)-(b-2)]]$$

Now, $\frac{df}{dx}=0 \Rightarrow x=\frac{a-1}{a+b-2}$

and $\left.\frac{d^2f}{dx^2}\right]_{x=\frac{a-1}{a+b-2}} < 0$

Hence $\frac{a-1}{a+b-2}$ is mode of distribution.

VI. (a) Let X be N (μ, σ) such that P (X < 89) = 0.90 and P(X < 94) = 0.94. Find μ and σ^2.

Sol. Given X ~ N (μ, σ^2)

$\therefore$ Standard Normal Variate $Z=\frac{X-\mu}{\sigma}$

In normal distribution we know 50% items are above mean and 50% items below mean.

(a) 90% items are below 89

$\therefore Z_1$ corresponding to 89 is $Z_1=\frac{89-\mu}{\sigma}$

$\therefore$ P $(0 < Z < Z_1) = .40$

$\therefore$ Z_1 corresponding to probability .40 is 1.28

$\frac{89-\mu}{\sigma}=1.28$ ..... (1)

(b) 95% items are below 94

$\therefore Z_1$ corresponding to 94 is $Z_1=\frac{94-\mu}{\sigma}$

$\therefore$ P $(0 < Z < Z_1) = .45$

$\therefore$ Z_1 corresponding to probability .45 is 1.65

$\frac{94-\mu}{\sigma}=1.65$ ..... (2)

Solving (1) and (2)

$\sigma = 13.51$ (approx)

$\mu = 71.73$ (approx)

Hence $\mu = 71.73$ and $\sigma^2 = 182.52$

(b) The joint probability distribution of random variable X and Y is given by:
P(X = 0, Y = 0) = P(X = 0, Y = 1) = P(X =1, Y = –1) and
P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X =1, Y = –1) = 1
Find:
(i) Marginal distributions of X and Y.
(ii) Condition probability distribution of X given Y = 0, where X and Y take value –1, 0 and 1.

Sol. 6 (b)

Let P(X = 0, Y = 0) = P (X = 0, Y = 1) = P(X = 1, Y = –1) = K

Also P(X = 0, Y = 0) + P (X = 0, Y = 1) + P(X = 1, Y = –1) = 1

$\Rightarrow$ K = ⅓

Therefore we have the following table.

Y \ X	–1	0	1	Marginal Y
–1	–	–	⅓	⅓
0	–	⅓	–	⅓
1	–	⅓	–	⅓
Marginal (X)	0	⅔	⅓	1

Thus Marginal distribution of X is :

X	–1	0	1
P(X)	0	⅔	⅓

Marginal distribution of Y is :

X	–1	0	1
P(Y)	⅓	⅓	⅓

Conditional distribution of X given Y = 0 is

$$P(X=-1, Y=0) = \frac{P(X=-1, Y=0)}{P(Y=0)} = \frac{0}{1/3} = 0$$

$$P(X=0, Y=0) = \frac{P(X=0, Y=0)}{P(Y=0)} = \frac{1/3}{1/3} = 1$$

$$P(X=1, Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{0}{1/3} = 0$$

The conditional distribution of X given Y = 0 is

X	–1	0	1
P(X = x | Y = 0)	0	1	0

VII. (a) Two random variables X and Y have the point p.d.f. :

$$f(x, y) = \begin{cases} 2-x-y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find:

(iii) Marginal probability density function of X and Y.
(iv) Condition density functions
(v) Var (X) and Var (Y)
(vi) Cov (X, Y).

Sol. (i) We have $f(x, y) = \begin{cases} 2-x-y, & \text{if } 0 \le x \le 1, 0 \le y \\ 0 & , \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 (2-x-y)dy = \frac{3}{2} - x$$

$\therefore$ Marginal p.d.f. of X is

$$f_x(x) = \begin{cases} \frac{3}{2} - x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly $f_y(y) = \begin{cases} \frac{3}{2} - y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) Conditional density function of X given Y = y is

$$f(x \mid y) = \frac{f(x, y)}{f_x(y)} = \frac{2-x-y}{\frac{3}{2} - y}, 0 \le x \le 1.$$

Conditional density function of Y given X = x is

$$f(y \mid x) = \frac{f(x, y)}{f_y(x)} = \frac{2-x-y}{\frac{3}{2} - x}, 0 \le y \le 1.$$

(iii) $E(x) = \int_0^1 x f_x(x)dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \frac{5}{12}$

$$E(x) = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx = \left[\frac{3}{6}x^3 - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}.$$

Similarly, $V(Y) = \frac{11}{144}$.

(iv) $E(XY) = \int_0^1\int_0^1 xy(2-x-y)dx\, dy$

$$= \int_0^1 \left[x^2y - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 x\left(\frac{2}{3}y - \frac{1}{2}y^2\right)dy = \left[\frac{y^2}{3} - \frac{y^3}{6}\right]_0^1 = \frac{1}{6}.$$

$$\therefore \quad Cov(X,Y) = E(XY) - E(X)\,E(Y) = \frac{1}{6} - \frac{5}{12} \times \frac{5}{12} = -\frac{1}{144}$$

(b) Let X and Y have a bivariate normal distribution with parameters $\mu_x = 20$, $\mu_y = 40$, $\sigma_x^2 = 9$, $\sigma_x^2 = 4$ and $\rho = 0.6$. Find the shortest interval for which 0.90 is the conditional probability that Y is in this interval, given that X = 22.

Sol. Let a < Y < b be the shortest interval for which
P (a < Y < b | X = 22) = 0.90.
We know that if X and Y are bivariate normal distribution, then

$$Y \mid X \text{ is } N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1-\rho^2)\right)$$

$$\therefore \quad Y \mid X = 22 \sim N\left(40 + 0.6\frac{2}{3}(22 - 20),\ 4(1 - 0.36)\right)$$

$$\therefore \quad \sim N\left(\frac{204}{5}, \frac{64}{25}\right).$$

Now, P (a< Y < b | X = 22) = 0.90

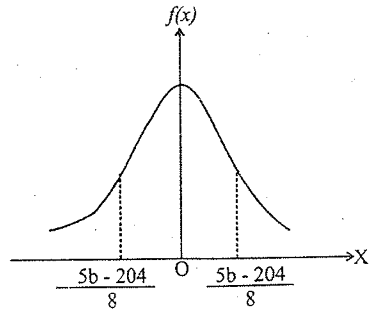
$$\Rightarrow \quad \left(\frac{a - \frac{204}{5}}{\frac{8}{5}} < \frac{y - \frac{204}{5}}{\frac{8}{5}} < \frac{b - \frac{204}{5}}{\frac{8}{5}}\right) = 0.90$$

$$\Rightarrow \quad \Phi\left(\frac{5b - 204}{8}\right) - \Phi\left(\frac{5a - 204}{8}\right) = 0.90$$

Now, the interval (a, b) will be shortest when it is symmetric about x = 0 so that

$$\frac{5b - 204}{8} = -\frac{5b - 204}{8}.$$

$$\therefore \quad \Phi\left(\frac{5a - 204}{8}\right) - \Phi\left(-\frac{5b - 204}{8}\right) = 0.90$$



$$\Rightarrow \quad \Phi\left(\frac{5b-204}{8}\right)-\left[1-\Phi\left(\frac{5b-204}{8}\right)\right]=0.90$$

$$\Rightarrow \quad \Phi\left(\frac{5b-204}{8}\right)=0.85$$

$$\Rightarrow \quad \frac{5b-204}{8}=1.645 \text{ and } \frac{5b-204}{8}=-1.645$$

$$\Rightarrow \quad b=\frac{1.645\times 8+204}{5} \text{ and } a=-\frac{1.645\times 8+204}{5}$$

$\Rightarrow$ $b = 43.432$ and $a = 38.168$.

Hence the required shortest interval is (38.168, 43.432).

VIII. (a) If X and Y are independent exponential distribution, then find the probability density function of $\frac{X+Y}{2}$.

Sol. Since X and Y are exponential variates, therefore their p.d.f.s. are given by

$$f_x(x)=\begin{cases} e^{-x}, & 0<x<\infty \\ 0, & \text{elsewhere}\end{cases} \text{ and } f_y(Y)=\begin{cases} e^{-x}, & 0<y<\infty \\ 0, & \text{elsewhere}\end{cases}$$

Since X and Y are independent, therefore their joint p.d.f.s. is given by

$$f_x(x,y)=f_x(x)f_y(y)=\begin{cases} e^{-(x+y)}, & 0<x<\infty,\ 0<y<\infty \\ 0, & \text{elsewhere}\end{cases}$$

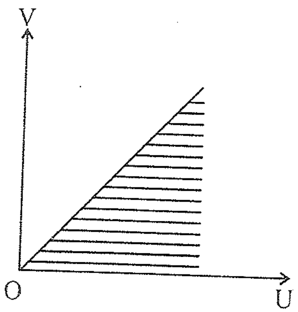
Consider the transformation $y=\frac{x+y}{2}$ and $v = y$.

Then $x = 2u - v$ and $y = v$.

$$\therefore \quad J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix} 2 & 0 \\ -1 & 1\end{vmatrix}=2.$$

Now, $y>0 \Rightarrow v>0$

and $x>0 \Rightarrow 2u-v>0 \Rightarrow v<2u$.



$\therefore$ Joint p.d.f. of U and V is given by

$$g(u,v)=f(x,y)\,|J|=f(2u-v,v)\,|J|$$

$$\begin{cases} 2e^{-2u}, & 0<v<2u<\infty \\ 0, & \text{elsewhere}\end{cases}$$

Now, $\int_{-\infty}^{\infty} g(u,v)dv=\int_0^{2u} 2e^{-2u}dv=4ue^{-2u}$

$\therefore$ Marginal distribution of U is given by

$$g_u(u)=\begin{cases} 4ue^{-2u}, & 0<u<\infty \\ 0, & \text{elsewhere}\end{cases}$$

(b) Prove that mean of F – distribution is $\frac{r_2}{r_2-2}$, $r_2>2$.

Sol. Let W has F-distribution, then p.d.f. of W is given by

$$f(w)=\begin{cases} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} \dfrac{w^{\frac{r_1}{2}-1}}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)\left(1+\frac{r_1}{r_2}w\right)^{\frac{r_1+r_2}{2}}}, & 0<w<\infty \\ 0, & \text{elsewhere}\end{cases}$$

Now, mean $= E(W) = \int_{-\infty}^{\infty} wf(w)dw$

$$=\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}\frac{1}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)}\int_0^{\infty}\frac{w^{\frac{r_1}{2}}}{\left(1+\frac{r_1}{r_2}w\right)^{\frac{r_1+r_2}{2}}}dw$$

$$=\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}\frac{1}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)}\int_0^{\infty}\frac{\left(\frac{r_2}{r_1}t\right)^{\frac{r_1}{2}}}{(1+t)^{\frac{r_1+r_2}{2}}}\left(\frac{r_2}{r_1}dt\right) \text{ putting } \frac{r_1}{r_2}w=t$$

so that $dw=\frac{r_2}{r_1}dt$

$$=\frac{r_1}{r_2}\frac{1}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)}\int_0^{\infty}\frac{t^{\frac{r_1}{2}}}{(1+t)^{\frac{r_1+r_2}{2}}}dt$$

$$=\left(\frac{r_2}{r_1}\right)\frac{1}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)}B\left(\frac{r_1}{2}+1,\frac{r_2}{2}-1\right)$$

$$=\frac{r_2}{r_1}\frac{\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}\frac{\Gamma\left(\frac{r_1}{2}+1\right)\Gamma\left(\frac{r_2}{2}-1\right)}{\Gamma\left(\frac{r_1+r_2}{2}\right)}$$

$$= \frac{r_2}{r_1} \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\left(\frac{r_2}{2}-1\right)\Gamma\left(\frac{r_2}{2}-1\right)} \frac{r_1}{2} \Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}-1\right)$$

$$= \frac{r_2}{r_1} \frac{1}{\frac{r_2}{2}-1} \cdot \frac{r_1}{2}$$

$$= \frac{r_2}{r_2 - 2}.$$

Part - A

1

RIEMANN INTEGRATION

1. Prove that every monotonically increasing function on a closed interval is Riemann integrable. (September 2013, April 2010, September 2009)

Sol. Let $f(x)$ be a monotonic function on [a, b]

Assume that f is monotonically increasing function i.e., for all $x', x'' \in [a,b]$,

$x' < x'' \Rightarrow f(x') < f(x'')$

Let $\varepsilon > 0$, however small.

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of [a, b] with

$$\|P\| < \frac{\varepsilon}{f(b) - f(a) + 1} > 0 \qquad [\because f(b) \geq f(a)]$$

$\because f$ is monotonically increasing on $[x_{i-1}, x_i]$

$\therefore\ m_i = f(x_{i-1}), M_i = f(x_i)$ for i = 1, 2, 3, ………… , n

Now $\text{U}(\text{P}, f) - \text{L}(\text{P}, f)$

$$= \sum_{i=1}^{n} M_i \delta_i - \sum_{i=1}^{n} m_i \delta_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \delta_i$$

$$\leq \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\} \frac{\varepsilon}{f(b) - f(a) + 1}$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}$$

$$=\frac{\varepsilon}{f(b)-f(a)+1}\{(f(x_1)-f(x_0))+(f(x_2)-f(x_1))$$
$$+\dots\dots+(f(x_n)-f(x_{n-1}))\}$$
$$=\frac{\varepsilon}{f(b)-f(a)+1}\left[f(x_n)-f(x_0)\right]$$
$$=\frac{\varepsilon}{f(b)-f(a)+1}\left[f(b)-f(a)\right]$$
$$=\frac{f(b)-f(a)}{f(b)-f(a)+1}\varepsilon$$

$\therefore U(P,f)-L(P,f)<\varepsilon$

$\Rightarrow f$ is R-integrable.

Similarly when f(x) is monotonically decreasing on [a, b], then taking

$$\|P\|<\frac{\varepsilon}{f(b)-f(a)+1}, m=f(x_i),\ M_i=f(x_{i-1}),$$

We can prove the required result.

2. Prove that $\sqrt{\frac{3}{2}}\le\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{\sqrt{2}}{6}$. **(September 2013)**

Sol. Let $f(x)=\frac{\sin x}{x}$

$$\therefore f'(x)=\frac{x\cos x-\sin x.1}{x^2}=\frac{\cos x(x-\tan x)}{x^2}\le 0\forall x\in\left[\frac{\pi}{4},\frac{\pi}{3}\right]$$
$$\left\{\because \tan x>x,\ \cos x\ge 0,\ x^2>0\forall x\in\left[\frac{\pi}{4},\frac{\pi}{3}\right]\right\}$$

$\therefore f$ is monotonically decreasing in $\left[\frac{\pi}{4},\frac{\pi}{3}\right]$

$$\therefore m=g.l.b.\ \left\{f(x):\frac{\pi}{4}\le x\le\frac{\pi}{3}\right\}$$
$$=g.l.b.\ \left\{\frac{\sin x}{x}:\frac{\pi}{4}\le x\le\frac{\pi}{3}\right\}$$

$$=\frac{\sin\frac{\pi}{3}}{\frac{\pi}{3}}=\frac{\frac{\sqrt{3}}{2}}{\frac{\pi}{3}}=\frac{3\sqrt{3}}{2\pi}$$

and $M=l.u.b.\ \left\{f(x):\frac{\pi}{4}\le x\le\frac{\pi}{3}\right\}$

$$=l.u.b.\ \left\{\frac{\sin x}{x}:\frac{\pi}{4}\le x\le\frac{\pi}{3}\right\}$$
$$=\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}}=\frac{\frac{1}{\sqrt{2}}}{\frac{\pi}{4}}=\frac{2\sqrt{2}}{\pi}$$

$\therefore$ by First Mean Value Theorem,

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a)$$
$$\therefore \frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\le\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$
$$\therefore \frac{3\sqrt{3}}{2\pi}\times\frac{\pi}{12}\le\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{2\sqrt{2}}{\pi}\times\frac{\pi}{12}$$
$$\therefore \frac{\sqrt{3}}{8}\le\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin x}{x}dx\le\frac{\sqrt{2}}{6}.$$

3. Prove that a continuous function *f* on [a, b] is Riemann-integrable.
(April 2013, April 2009)

Sol. Let f be a function which is continuous on the closed interval [a, b]

$\therefore$ f is uniformly continuous on [a, b]

$\Rightarrow$ for a given $\varepsilon>0$, however small, we can find a positive real number δ such that

$$|f(x')-f(x'')|<\frac{\varepsilon}{b-a} \quad (1)$$

for all $x', x''\in$ [a, b] satisfying $|x'-x''|<\delta$

Let P = {a = x_0, x_1, x_2, ….., x_n = b} be any partition of [a, b] with $\|P\|<\delta$

Since f is continuous on [a, b] therefore, it is continuous on $[x_{i-1}, x_i]$ for i = 1, 2, 3,…., n.

Let $m_i = f(c_i), M_i = f(d_i)$

$\therefore \|P\| < \delta \quad \Rightarrow \quad |x_i - x_{i-1}| < \delta$

$\Rightarrow |d_i - c_i| < \delta \qquad \{\because c_i, d_i \in [x_{i-1}, x_i]\}$

$\therefore M_i - m_i = |M_i - m_i| \qquad [\because M_i \geq m_i]$

$= |f(d_i) - f(c_i)|$

$< \frac{\varepsilon}{b-a} \qquad [\because \text{of } (1)]$

$\Rightarrow M_i - m_i < \frac{\varepsilon}{b-a} \qquad$ (2)

for i = 1, 2, 3, ……, n.

Now $U(P,f) - L(P,f) = \sum_{i=1}^{n} M_i\delta_i - \sum_{i=1}^{n} m_i\delta_i$

$= \sum_{i=1}^{n} (M_i - m_i)\delta_i$

$< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a}\right)\delta_i \qquad [\because \text{of } (2)]$

$= \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \delta_i$

$= \frac{\varepsilon}{b-a}(b-a) \qquad \left[\because \sum_{i=1}^{n} \delta_i = b-a\right]$

$\therefore$ U(P, f) – L(P, f) < ε

$\Rightarrow$ f is R-integrable.

4. **Evaluate:** **(April 2013)**

(i) L(P, f) where $f(x) = x^2$ and $P = \left\{\frac{-3}{2}, \frac{-1}{2}, \frac{1}{4}, 1\right\}$

(ii) U(P, f) where $f(x) = \sin x$ and $P = \left\{0, \frac{\pi}{4}, \frac{2\pi}{3}, \pi\right\}$.

Sol. (i) Here $P = \left\{\frac{-3}{2}, \frac{-1}{2}, \frac{1}{4}, 1\right\}$ is divided into 3 sub-intervals

$I_1 = \left[\frac{-3}{2}, \frac{-1}{2}\right], I_2 = \left[\frac{-1}{2}, \frac{1}{4}\right], I_3 = \left[\frac{1}{4}, 1\right]$ such that

$\delta_1 = \frac{-1}{2} + \frac{3}{2} = 1,\ \delta_2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4},\ \delta_3 = 1 - \frac{1}{4} = \frac{3}{4}$

Let m_i, M_i be g.l.b's and l.u.b's of f in I_i for i = 1, 2, 3,

Now for I_1 $-\frac{3}{2} \leq x \leq -\frac{1}{2} \Rightarrow \frac{1}{4} \leq x^2 \leq \frac{9}{4} \Rightarrow m_1 = \frac{1}{4}, M_1 = \frac{9}{4}$

for I_2, $-\frac{1}{2} \leq x \leq \frac{1}{4} \Rightarrow \frac{1}{16} \leq x^2 \leq \frac{1}{4} \Rightarrow m_2 = \frac{1}{16}, M_2 = \frac{1}{4}$

and for I_3, $-\frac{1}{2} \leq x \leq 1 \Rightarrow \frac{1}{16} \leq x^2 \leq 1 \Rightarrow m_3 = \frac{1}{16}, M_3 = 1$

Hence L(P, f) = $m_1\delta_1 + m_2\delta_2 + m_3\delta_3 = \left(\frac{1}{4}\right)(1) + \left(\frac{1}{16}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{16}\right)\frac{3}{4}$

$= \frac{16+3+3}{(16)(4)} = \frac{22}{64} = \frac{11}{32}$

(ii) Here $P = \left\{0, \frac{\pi}{4}, \frac{2\pi}{3}, \pi\right\}$ is divided into 3 sub-intervals

$I_1 = \left[0, \frac{\pi}{4}\right],\ I_2 = \left[\frac{\pi}{4}, \frac{2\pi}{3}\right],\ I_3 = \left[\frac{2\pi}{3}, \pi\right]$ such that

$\delta_1 = \frac{\pi}{4} - 0 = \frac{\pi}{4},\ \delta_2 = \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12},\ \delta_3 = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

Let m_i, M_i be g.l.b's and l.u.b's of f in I_i for i = 1, 2, 3

for I_1, $0 \leq x \leq \frac{\pi}{4} \Rightarrow 0 \leq \sin x \leq \frac{1}{\sqrt{2}} \Rightarrow m_1 = 0, M_1 = \frac{1}{\sqrt{2}}$

for I_2, $\frac{\pi}{4} \leq x \leq \frac{2\pi}{3} \Rightarrow \frac{1}{\sqrt{2}} \leq \sin x \leq \frac{\sqrt{3}}{2} \Rightarrow m_2 = \frac{1}{\sqrt{2}}, M_2 = \frac{\sqrt{3}}{2}$

for I_3, $\frac{2\pi}{3} \leq x \leq \pi \Rightarrow 0 \leq \sin x \leq \frac{1}{\sqrt{2}} \Rightarrow m_3 = 0, M_3 = \frac{1}{\sqrt{2}}$

Hence $U(P,f) = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) + \frac{\sqrt{3}}{2}\left(\frac{5\pi}{12}\right) + \frac{1}{\sqrt{2}}\left(\frac{\pi}{3}\right)$

$= \frac{6\pi + 5\sqrt{2}\pi + 8\pi}{24\sqrt{2}} = \frac{(14+5\sqrt{2})\pi}{24\sqrt{2}}$

$= \frac{(10+14\sqrt{2})\pi}{48}$

$= \frac{(5+7\sqrt{2})\pi}{24}$

5. **Let $f(x)=\sin x$ on $\left[0,\frac{\pi}{2}\right]$. Evaluate $\int_0^{\frac{\pi}{2}} f dx$ and $\int_0^{\frac{\pi}{2}} f dx$ by dividing $\left[0,\frac{\pi}{2}\right]$ into n equal parts and show that $f \in R(x)$ on $\left[0,\frac{\pi}{2}\right]$.** **(September 2012)**

Sol. Here $f(x)=\sin x$

Let $P=\left\{0,\frac{\pi}{2n},\frac{2\pi}{2n},\frac{3\pi}{2n},\ldots,\frac{n\pi}{2n}\right\}$

or $P=\left\{x_0=0,x_1,x_2,\ldots,x_n=\frac{\pi}{2}\right\}$ be a partition of $\left[0,\frac{\pi}{2}\right]$ such that

$x_i=\frac{i\pi}{2n}$ for $i=1,2,3,\ldots,n$

Now $\delta_i=\frac{\pi}{2n}$ for each $i=1,2,3,\ldots,n$

Since $f(x)=\sin x$ is monotonically increasing in $\left[0,\frac{\pi}{2}\right]$

$\therefore\ m_i=f(x_{i-1})=\sin\left\{\frac{(i-1)\pi}{2n}\right\}$

and $M_i=f(x_i)=\sin\left(\frac{i\pi}{2n}\right)$

$$L(P,f)=\sum_{i=1}^{n} m_i\delta_i$$

$$=\sum_{i=1}^{n}\left[\sin\left\{\frac{(i-1)\pi}{2n}\right\}\cdot\frac{\pi}{2n}\right]$$

$$=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\left\{\frac{(i-1)\pi}{2n}\right\}$$

$$=\frac{\pi}{2n}\left[0+\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\sin\frac{3\pi}{2n}+\ldots+\sin\left\{\frac{(n-1)\pi}{2n}\right\}\right]$$

$$=\frac{\pi}{2n}\left[\frac{\sin\frac{(n-1)\pi}{2.2n}\sin\left\{\frac{\pi}{2n}+\frac{(n-2)\pi}{2.2n}\right\}}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{2n}\right)}\right]$$

$$\left[\because \sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots \text{ upto n terms} =\frac{\sin\frac{n\beta}{2}\sin\left(\alpha+\overline{n-1}\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}\right]$$

$$=\frac{\frac{2\sin\frac{(n-1)\pi}{4\pi}\sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}}}{\frac{\pi}{4n}}=\frac{\frac{\sqrt{2}\sin\frac{(n-1)\pi}{4n}}{\sin\frac{\pi}{4n}}}{\frac{\pi}{4n}}$$

$$U(P,f)=\sum_{i=1}^{n}M_i\delta_i=\sum_{i=1}^{n}\left(\sin\frac{i\pi}{2n}\right)\cdot\frac{\pi}{2n}$$

$$=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\frac{i\pi}{2n}$$

$$=\frac{\pi}{2n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\sin\frac{3\pi}{2n}+\ldots+\sin\frac{n\pi}{2n}\right]$$

$$=\frac{\pi}{2n}\cdot\frac{\sin\left(\frac{n\frac{\pi}{2n}}{2}\right)\sin\left[\frac{\pi}{2n}+(n-1)\frac{\pi}{2.2n}\right]}{\sin\left(\frac{\pi}{2.2n}\right)}$$

$$=\frac{\pi}{2n}\cdot\frac{\sin\frac{\pi}{4}\sin\frac{(n+1)\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$=\frac{\sqrt{2}\sin\frac{(n+1)\pi}{4n}}{\left(\frac{\sin\frac{\pi}{4n}}{\frac{\pi}{4n}}\right)}$$

$$\therefore\ \int_0^{\frac{\pi}{2}} f(x)\,dx=\underset{\|P\|\to 0}{\text{Lt}}\ L(P,f)$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{2}\sin\left\{\frac{(n-1)\pi}{4n}\right\}}{\left(\frac{\sin\frac{\pi}{4n}}{\frac{\pi}{4n}}\right)}$$

$$= \frac{\sqrt{2}.\frac{1}{\sqrt{2}}}{1} \qquad \left[\because \underset{n\to\infty}{\text{Lt}} \sin\frac{(n-1)\pi}{4n} = \frac{1}{\sqrt{2}}\right]$$

$$= 1$$

$$\int_0^{\frac{\pi}{2}} f(x)\,dx = \underset{\|P\|\to 0}{\text{Lt}} U(P,f)$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{2}\sin\left\{\frac{(n+1)\pi}{4n}\right\}}{\left(\frac{\sin\frac{\pi}{4n}}{\frac{\pi}{4n}}\right)}$$

$$= \frac{\sqrt{2}.\frac{1}{\sqrt{2}}}{1} \qquad \left[\because \underset{n\to\infty}{\text{Lt}} \sin\frac{(n+1)\pi}{4n} = \frac{1}{\sqrt{2}}\right]$$

$$\therefore \underline{\int_0^{\frac{\pi}{2}}} f(x)\,dx = \overline{\int_0^{\frac{\pi}{2}}} f(x)\,dx = 1$$

$\Rightarrow f(x)$ is R-integrable and $\int_0^{\frac{\pi}{2}} f(x)\,dx = 1$

Hence $\int_0^{\frac{\pi}{2}} \sin x\,dx = 1$

6. **If f is bounded and Integrable in [a, b] then $|f|$ is also bounded and Integrable in [a, b]. Moreover $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$. But converse is not always true, give suitable example.** **(September 2012, April 2012)**

Sol. First Part:

Since f is R-integrable on [a, b]

$\therefore f$ is bounded on [a, b] and the set D of the points of discontinuity of f in [a, b] is of measure zero.

As f is bounded on [a, b]

$\therefore$ there exists a positive number k such that

$$|f(x)| \le k \quad \forall x \in [a,b]$$

$$\Rightarrow |f|(x) \le k \quad \forall x \in [a,b]$$

$\Rightarrow |f|$ is bounded on [a, b]

We know that if f is continuous at any point of [a, b], then $|f|$ is also continuous at that point. Hence $|f|$ is continuous at every point of [a, b] – D. Also $|f|$ can be continuous at some points of D. Therefore, the set of the points of discontinuity of $|f|$ in [a, b] is a subset of D, which is of measure zero.

As every subset of a set of measure zero is itself a set of measure zero.

$\therefore$ the set of the points of discontinuity of $|f|$ in [a, b] is of measure zero

$\therefore |f|$ is R-integrable on [a, b]

Now $f(x) \le |f(x)| \forall x \in [a,b]$

$$\Rightarrow f(x) \le |f|(x) \forall x \in [a,b]$$

$$\Rightarrow \int_a^b f(x)\,dx \le \int_a^b |f|(x)\,dx \qquad (1)$$

Also $(-f)(x) = -f(x) \le |f(x)| \forall x \in [a,b]$

$$\Rightarrow \int_a^b (-f)(x)\,dx \le \int_a^b |f|(x)\,dx$$

$$\Rightarrow -\int_a^b f(x)\,dx \le \int_a^b |f|(x)\,dx \qquad (2)$$

From (1) and (2), we get

$$\max.\left\{\int_a^b f(x)\,dx, -\int_a^b f(x)\,dx\right\} \le \int_a^b |f|(x)\,dx$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f|(x)\,dx$$

Second part

Consider the function f defined on [0, 1] by

$$f(x)=\begin{cases}1, & \text{if x is rational}\\ -1, & \text{if x is irrational}\end{cases}$$

Now $-1\le f(x)\le 1\ \forall x\in[0,1]$

$\therefore$ f is bounded in [0, 1]

Let $P=\{0=x_0,x_1,x_2,............,x_n=1\}$ be any partition

For any sub-interval $[x_{i-1},x_i]$, we have

$m_i=-1$, $M_i=1$ for $i=1, 2, 3, ..., n$

[$\because$ every sub-interval contains rational as well as irrational points]

$$L(P,f)=\sum_{i=1}^{n} m_i\delta_i=(-1)(\delta_1)+(-1)(\delta_2)+....+(-1)(\delta_n)$$

$$=-(\delta_1+\delta_2+.......+\delta_n)=-(1-0)=-1$$

$$U(P,f)=\sum_{i=1}^{n} M_i\delta_i=1\cdot\delta_1+1\cdot\delta_2+....+1\cdot\delta_n$$

$$=(\delta_1+\delta_2+.......+\delta_n)=1-0=1$$

Now $\underline{\int_0^1} f(x)\,dx=\underset{\|P\|\to\infty}{\text{Lt}}\ L(P,f)=-1$

and $\overline{\int_0^1} f(x)\,dx=\underset{\|P\|\to\infty}{\text{Lt}}\ U(P,f)=1$

$\therefore\ \underline{\int_0^1} f(x)\,dx\ne\overline{\int_0^1} f(x)\,dx$

$\Rightarrow$ f is not R-integrable

Now $|f|(x)=|f(x)|=1\ \forall x\in[0,1]$

$\Rightarrow |f|$ is constant function on [0, 1]

$\Rightarrow |f|$ is R-integrable on [0, 1]

7. **Prove that** $\dfrac{1}{\sqrt{2}}\le\displaystyle\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-x^4}}dx\le\sqrt{\frac{2}{3}}$. **(April 2012)**

Sol. Consider the function $f(x)=\dfrac{1}{\sqrt{1-x^4}}, x\in\left[0,\dfrac{1}{\sqrt{2}}\right]$,

Now f is strictly increasing on $\left[0,\dfrac{1}{\sqrt{2}}\right]$ and

$$\therefore\ m=g.l.b.\left\{f(x):x\in\left[0,\frac{1}{\sqrt{2}}\right]\right\}$$

$$=g.l.b.\left\{\frac{1}{\sqrt{1-x^4}}:x\in\left[0,\frac{1}{\sqrt{2}}\right]\right\}=\frac{1}{\sqrt{1-0}}=1$$

$$M=l.u.b.\left\{f(x):x\in\left[0,\frac{1}{\sqrt{2}}\right]\right\}$$

$$=l.u.b.\left\{\frac{1}{\sqrt{1-x^4}}:x\in\left[0,\frac{1}{\sqrt{2}}\right]\right\}$$

$$=\frac{1}{\sqrt{1-\frac{1}{4}}}=\frac{2}{\sqrt{3}}$$

By first mean value theorem

$$m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$$

$$\therefore\ 1\left(\frac{1}{\sqrt{2}}-0\right)\le\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-x^4}}dx\le\frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}-0\right)$$

$$\Rightarrow\ \frac{1}{\sqrt{2}}\le\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-x^4}}dx\le\sqrt{\frac{2}{3}}$$

8. **A function f is defined as** $f(x)=\begin{cases}\mathbf{k\ for\ }x\ne 0\\ \mathbf{0\ for\ }x=0\end{cases} k>0$ **(September 2011)**

Show f is R-integrable on [–1, 1] and find $\displaystyle\int_{-1}^{1} f dx$.

Sol. Clearly $0\le f(x)\le k \quad \forall x\in[-1,1]$

Hence f is bounded on $[-1,1]$

Also f is continuous everywhere on $[-1,1]$ except for x = 0

$\Rightarrow$ f has finite no. of discontinuity

Hence f is integrable on $[-1,1]$

Let $P=\{-1=x_0,x_1,........,x_n=0=y_0,y_1,.......y_n=1\}$ be a partition of $[-1,1]$ which divides it into 2n equal parts each of length $\frac{1}{n}$ s.t. $\|p\|\to 0$ as $n\to\infty$

Also $x_i=-1+\frac{i}{n}$; $i=1,2,........,n$

and $y_i=0+\frac{i}{n}$; $i=1,2,........,n$

$$\therefore S(P,f)=\sum_{i=1}^{n}f(t_i)(x_i-x_{i-1})+\sum_{i=1}^{n}f(t_i')(y_i-y_{i-1})$$

$$=\sum_{i=1}^{n}f(t_i)\left(\frac{1}{n}\right)+\sum_{i=1}^{n}f(t_i')\left(\frac{1}{n}\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left(f(t_i)+f(t_i')\right) \quad \text{where } t_i\in[x_{i-1},x_i]\ t_i'\in[y_{i-1},y_i]$$

Taking $t_i=x_i$ and $t_i'=y_i$

s.t. $f(t_i)=f(x_i)=\begin{cases}k & \text{if } i\neq n\\ 0 & \text{if } i=n\end{cases}$

and $f(t_i')=f(y_i)=\begin{cases}0 & \text{if } i=0\\ k & \text{if } i\neq 0\end{cases}$

$$\Rightarrow S(P,f)=\frac{1}{n}\left[\sum_{i=1}^{n-1}f(x_i)+f(x_n)+\sum_{i=1}^{n}f(y_i)\right]$$

$$=\frac{1}{n}\left[\sum_{i=1}^{n-1}k+0+\sum_{i=1}^{n}k\right]=\frac{1}{n}\left[(n-1)k+nk\right]=\frac{(2n-1)k}{n}$$

$$\therefore \lim_{\|p\|\to 0}S(P,f)=\lim_{n\to\infty}\frac{(2n-1)k}{n}=2k$$

$$\Rightarrow \int_{-1}^{1}f(x)=2k$$

9. **State first mean value theorem of integral calculus and use it to find ξ if $\int_1^3 f(x)dx=2f(\xi)$ where f(x) = x² – 2x + 1.** **(September 2011)**

Sol. **First mean value theorem of Integral calculus:**
Let f be a continuous function on [a, b], then there exists a number

$c\in[a,b]$ s.t. $\int_a^b f(x)dx=f(c)(b-a)$

Now $f(x)=x^2-2x+1=(x-1)^2$

Since f is continuous on [1, 3]

$$\therefore \int_1^3 f(x)dx=\int_1^3(x-1)^2dx=\left[\frac{(x-1)^3}{3}\right]_1^3=\frac{8}{3}-0=\frac{8}{3}$$

By first mean value theorem of integral calculus

$\int_a^b f(x)dx=f(\xi)(b-a)$ wehere $\xi\in[a,b]$

$$\Rightarrow \int_1^3 f(x)dx=\frac{8}{3}=f(\xi)(3-1)$$

$$\Rightarrow f(\xi)=\frac{4}{3} \quad \text{or} \quad (\xi-1)^2=\frac{4}{3}$$

$$\Rightarrow \xi-1=\pm\frac{2}{\sqrt{3}}$$

$\Rightarrow \xi=1\pm\frac{2}{\sqrt{3}}$ But $1-\frac{2}{\sqrt{3}}\notin[1,3]$

$\Rightarrow \xi=1+\frac{2}{\sqrt{3}}$ is required value

10. **State and prove Fundamental Theorem of integral calculus.** **(April 2011)**

Sol. **Statement:** If f is R-integrable on [a, b] and F is a function defined on [a, b] such that

$F'(x)=f(x) \quad \forall x\in[a,b]$, then

$\int_a^x f(x)dx=F(x)-F(a) \quad \forall x\in[a,b]$ and in particular

$\int_a^b f(x)dx=F(b)-F(a)$

Proof: Since f is R – integrable on [a, b]

$\therefore$ f is R – integrable [a, x] $\forall x\in[a,b]$

Let $P=\{a=x_0,x_1,x_2,............x_{n-1},x_n=x\}$ be any partition of [a, x] and $\delta_i=x_i-x_{i-1}$ for i = 1, 2,….., n.

$\because$ F is differentiable on [a, b]

$\therefore$ F is differentiable on $[x_{i-1}, x_i]$ for i = 1, 2, ..., n

By Lagrange's Mean Value Theorem, there exists point $\xi_i\,(i=1,2,.....,n)$ in $[x_{i-1}, x_i]$ such that

$$F(x_i)-F(x_{i-1})=(x_i-x_{i-1})F'(\xi_i)$$

$$\Rightarrow F(x_i)-F(x_{i-1})=\delta_i f(\xi_i) \qquad \{\because f(x)=F'(x)\,\forall x\in[a,b]\}$$

$$\Rightarrow \sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]=\sum_{i=1}^{n}\delta_i f(\xi_i)$$

$$\Rightarrow \{F(x_1)-F(x_0)\}+\{F(x_2)-F(x_1)\}+\{F(x_3)-F(x_2)\}$$
$$+.....+\{F(x_n)-F(x_{n-1})\}=\sum_{i=1}^{n}\delta_i f(\xi_i)$$

$$\Rightarrow F(x_n)-F(x_0)=\sum_{i=1}^{n}\delta_i f(\xi_i)$$

Let n $\to\infty$ so that $\|P\|\to 0$

$$\therefore\ F(x)-F(a)=\int_a^x f(x)\,dx$$

$$\left\{\because \int_a^x f(x)\,dx=\underset{n\to\infty}{\mathrm{Lt}}\sum_{i=1}^{n}\delta_i f(\xi_i) \text{ as } f \text{ is R-integrable on } [a,x]\right\}$$

In particular when x = b, we get,

$$F(b)-F(a)=\int_a^b f(x)\,dx$$

or $\displaystyle\int_a^b f(x)\,dx=F(b)-F(a)$

11. $f(x)=\dfrac{1}{x^2}$ **defined on** $[1,4]$, $P_1=\{1,2,3,4\}$ **and** $P_2=\left\{1,\dfrac{7}{4},\dfrac{5}{2},\dfrac{13}{4},4\right\}$ **are two partitions of [1, 4], Verify** $L(P_1,f)\le L(P_2,f)$. **(April 2011)**

Sol. $f(x)=\dfrac{1}{x^2}$

Consider $P_1=\{1,2,3,4\}$ divided in 3 sub intervals $I_1=[1,2]$, $I_2=[2,3]$, $I_3=[3,4]$, length of each sub interval = 1

$\therefore\ \delta_i=1$ for i = 1, 2, 3

Let m_i, M_i be g.*l*.b's and *l*.u.b's of f in I_i for i = 1, 2, 3

When $x\in I_1$ $\quad 1\le x\le 2 \Rightarrow \dfrac{1}{4}\le\dfrac{1}{x^2}\le 1 \Rightarrow m_1=\dfrac{1}{4}, M_1=1$

$x\in I_2$ $\quad 2\le x\le 3 \Rightarrow \dfrac{1}{9}\le\dfrac{1}{x^2}\le\dfrac{1}{4} \Rightarrow m_2=\dfrac{1}{9}, M_2=\dfrac{1}{4}$

$x\in I_3$ $\quad 3\le x\le 4 \Rightarrow \dfrac{1}{16}\le\dfrac{1}{x^2}\le\dfrac{1}{9} \Rightarrow m_3=\dfrac{1}{16}, M_3=\dfrac{1}{9}$

$$\Rightarrow\quad L(P,f)=m_1\delta_1+m_2\delta_2+m_3\delta_3=\frac{1}{4}+\frac{1}{9}+\frac{1}{16}=\frac{61}{144}=0.4236$$

Consider $P_2=\left\{1,\dfrac{7}{4},\dfrac{5}{2},\dfrac{13}{4},4\right\}$ divided in 4 sub intervals $I_1=\left[1,\dfrac{7}{4}\right]$, $I_2=\left[\dfrac{7}{4},\dfrac{5}{2}\right]$, $I_3=\left[\dfrac{5}{2},\dfrac{13}{4}\right]$ and $I_4=\left[\dfrac{13}{4},4\right]$ such that length of each subinterval $=\dfrac{3}{4}$, $\delta_i=\dfrac{3}{4}$ for i = 1, 2, 3, 4

When $x\in I_1$ $\quad 1\le x\le\dfrac{7}{4} \Rightarrow \dfrac{16}{49}\le\dfrac{1}{x^2}\le 1 \Rightarrow m_1=\dfrac{16}{49}, M_1=1$

$x\in I_2$ $\quad \dfrac{7}{4}\le x\le\dfrac{5}{2} \Rightarrow \dfrac{4}{25}\le\dfrac{1}{x^2}\le\dfrac{16}{49} \Rightarrow m_2=\dfrac{4}{25}, M_2=\dfrac{16}{49}$

$x\in I_3$ $\quad \dfrac{5}{2}\le x\le\dfrac{13}{4} \Rightarrow \dfrac{16}{169}\le\dfrac{1}{x^2}\le\dfrac{4}{25} \Rightarrow m_3=\dfrac{16}{169}, M_3=\dfrac{4}{25}$

$x\in I_4,$ $\quad \dfrac{13}{4}\le x\le\dfrac{4}{1} \Rightarrow \dfrac{1}{16}\le\dfrac{1}{x^2}\le\dfrac{16}{169} \Rightarrow m_4=\dfrac{1}{16}, M_4=\dfrac{16}{169}$

$$\Rightarrow\quad L(P_2,f)=m_1\delta_1+m_2\delta_2+m_3\delta_3+m_4\delta_4=\frac{3}{4}\left(\frac{16}{49}+\frac{4}{25}+\frac{16}{169}+\frac{1}{16}\right)=.4827$$

$$\Rightarrow\quad L(P_1,f)\le L(P_2,f)$$

12. **Define Lower and Upper Riemann Integrals of a bounded function on [a, b] and prove that lower Riemann integral cannot exceed the Upper Riemann integral.** **(September 2010)**

Sol. **First part**

The l.u.b. of the set {L (P, *f*) : P ∈ P [a, b]} is called the lower Reimann integral (or lower R-integral) of *f* on [a, b] and is denoted by the symbol $\underline{\int_a^b} f(x)\,dx$

$$\therefore\ \underline{\int_a^b} f(x)\,dx=l.u.b.\{L(P,f)\}$$

$$\underline{\int_a^b} f(x)\,dx=l.u.b.\{L(P,f):P \text{ is a partition of } [a,b]\}$$

Similarly the g.l.b. of the set $\{U(P,f): P \in P[a,b]\}$ is called upper Riemann integral (or upper R-integral) of f on [a, b] and is denoted by the symbol $\overline{\int_a^b} f(x)\,dx$

$\therefore \overline{\int_a^b} f(x)dx = g.l.b.\{U(P,f): P \text{ is a partition of } [a,b]\}$

Second Part

Let P_1, P_2 be any two partitions of [a, b]

$\therefore$ we have

$L(P_1, f) \le U(P_2, f)$ (1)

Keeping P_1 fixed, we observe from (1) that the number $L(P_1, f)$ is less than or equal to every number of the set $\{U(P_2, f) : P_2$ is a partition of [a, b]$\}$ of upper sums.

$\Rightarrow L(P_1, f)$ is a lower bound of the set of upper sums.

But $\overline{\int_a^b} f dx = g.l.b.\{U(P_2,f): P_2 \text{ is a partition of } [a,b]\}$

$\therefore L(P_1, f)$ is a lower bound of the set whose g.l.b. is $\overline{\int_a^b} f(x)\,dx$

$\Rightarrow L(P_1,f) \le \overline{\int_a^b} f dx$

[$\because$ a lower bound of set is always less than or equal to its greatest lower bound]

$\Rightarrow \overline{\int_a^b} f dx \ge L(P_1,f)$ where P_1 is any partition of $[a,b)$

$\Rightarrow \overline{\int_a^b} f dx \ge$ every member of $\{L(P_1,f): P_1 \text{ is a partition of } [a,b]\}$ whose *lub* is $\underline{\int_a^b} f dx$

$$\left[\because \underline{\int_a^b} f dx = l.u.b.\{L(P_1,f): P_1 \text{ is a partition of } [a,b]\}\right]$$

$\Rightarrow \overline{\int_a^b} f dx \ge \underline{\int_a^b} f dx$

[$\because$ an upper bound of a set $\ge$ the *l.u.b.* of the set]

$\Rightarrow \underline{\int_a^b} f dx \le \overline{\int_a^b} f dx$

13. Show by giving an example of a bounded function f such that f is not R-integrable but $|f|$ is R-integrable.

(September 2010, September 2011, April 2010)

Sol. Consider the function f defined on [0, 1] by

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ -1, & \text{if x is irrational} \end{cases}$$

Now $-1 \le f(x) \le 1 \;\; \forall x \in [0,1]$

$\therefore$ f is bounded in [0, 1]

Let $P = \{0 = x_0, x_1, x_2, \ldots\ldots, x_n = 1\}$ be any partition of [0, 1]

For any sub-interval $[x_{i-1}, x_i]$, we have

$m_i = -1$, $M_i = 1$ for i = 1, 2, 3, ..., n

[$\because$ every sub-interval contains rational as well as irrational points]

$L(P,f) = \sum_{i=1}^{n} m_i \delta_i = (-1)(\delta_1) + (-1)(\delta_2) + \ldots + (-1)(\delta_n)$

$= -(\delta_1 + \delta_2 + \ldots\ldots + \delta_n) = -(1-0) = -1$

and $U(P,f) = \sum_{i=1}^{n} M_i \delta_i = 1.\delta_1 + 1.\delta_2 + \ldots + 1.\delta_n$

$= (\delta_1 + \delta_2 + \ldots\ldots + \delta_n) = 1 - 0 = 1$

Now $\underline{\int_0^1} f(x)dx = \underset{\|P\|\to 0}{Lt} L(P,f) = -1$

and $\overline{\int_0^1} f(x)dx = \underset{\|P\|\to 0}{Lt} U(P,f) = 1$

$\therefore \underline{\int_0^1} f(x)dx \ne \overline{\int_0^1} f(x)dx$

$\Rightarrow$ f is not R-integrable

Now $|f|(x) = |f(x)| = 1 \;\; \forall x \in [0,1]$

$\Rightarrow |f|$ is constant function n [0, 1]

$\Rightarrow |f|$ is R-integrable on [0, 1]

14. If $f(x)=2x+3$, show that $f(x)$ is R – integrable on [0, 2] and $\int_0^2 f(x)dx=10$. **(September 2009)**

Sol. Here $x \in [0, 2]$

$\Rightarrow 0 \le x \le 2 \Rightarrow 0 \le 2x \le 4$

$\Rightarrow 3 \le 2x+3 \le 4+3 \Rightarrow 3 \le f(x) \le 7$

$\Rightarrow f$ is bounded on [0, 2].

Let P = $\left\{0, \frac{2}{n}, \frac{4}{n}, \ldots\ldots, \frac{2(i-1)}{n}, \frac{2i}{n}, \ldots\ldots\ldots, \frac{2n}{n}=2\right\}$ be partition of [0, 2]

Comparing it with P $=\{x_0=0, x_1, x_2, \ldots\ldots\ldots, x_n=2\}$, we have,

$x_i=\frac{2i}{n}, i=0,1,2,\ldots\ldots\ldots,n$

$\because f$ is monotonically increasing

$\therefore m_i=f(x_{i-1})=2x_{i-1}+3=2\left\{\frac{2(i-1)}{n}\right\}+3=\frac{4(i-1)}{n}+3$

and $M_i=f(x_i)=2x_i+3=2\left(\frac{2i}{n}\right)+3=\frac{4i}{n}+3$

$L(P,f)=\sum_{i=1}^{n} m_i\delta_i=\sum_{i=1}^{n}\left\{\frac{4(i-1)}{n}+3\right\}.\frac{2}{n}$

$=\sum_{i=1}^{n}\frac{8}{n^2}(i-1)+\sum_{i=1}^{n}\frac{6}{n}$

$=\frac{8}{n^2}[0+1+2+\ldots.+(n-1)]+\frac{6}{n}\times n$

$=\frac{8}{n^2}\left[\frac{n-1}{2}(1+n-1)\right]+6$

$=\frac{8}{n^2}\frac{n(n-1)}{2}+6=4\left(1-\frac{1}{n}\right)+6$

and $U(P,f)=\sum_{i=1}^{n} M_i\delta_i=\sum_{i=1}^{n}\left(\frac{4i}{n}+3\right).\frac{2}{n}$

$=\sum_{i=1}^{n}\frac{8}{n^2}i+\sum_{i=1}^{n}\frac{6}{n}$

$=\frac{8}{n^2}(1+2+3+\ldots.+n)+\frac{6}{n}\times n$

$=\frac{8}{n^2}\frac{n(n+1)}{2}+6=4\left(1+\frac{1}{n}\right)+6$

Now $\underline{\int_0^2} f(x)dx=\underset{\|P\|\to 0}{Lt}L(P,f)$

$=\underset{n\to\infty}{\text{Lt}}\left[4\left(1-\frac{1}{n}\right)+6\right]$ $\quad\left[\because n\to\infty \text{ as } \|P\|=\frac{2}{n}\to 0\right]$

$=4(1-0)+6=10$

and $\overline{\int_0^2} f(x)dx=\underset{\|P\|\to 0}{Lt}U(P,f)$

$=\underset{n\to\infty}{\text{Lt}}\left[4\left(1+\frac{1}{n}\right)+6\right]=4(1+0)+6=10$

$\because \underline{\int_0^2} f(x)dx=\overline{\int_0^2} f(x)dx=10$

$\therefore f\in R(x)$ on [0, 2] and $\int_0^2 f(x)dx=10$

i.e. $\int_0^2 (2x+3)dx=10$

15. Show that the function f defined by: **(April 2009)**

$f(x)=x^2, x\in[0,a], a>0$ is R-integrable on $[0,a]$ and $\int_0^a x^2dx=\frac{a^3}{3}$.

Sol. Let P = $\{0=x_0, x_1, x_2\ldots\ldots, x_{n-1}, x_n=a\}$ be any partition of [0, a] where $x_i=\frac{ai}{n}$ for i = 1, 2, ………, n

For any sub-interval $[x_{i-1}, x_i]$

$m_i=g.l.b.\ \{x^2 : x_{i-1}\le x\le x_i\}$

$=g.l.b.\ \left\{x^2 : \frac{i-1}{n}a\le x\le \frac{i}{n}a\right\}$

$=\left(\frac{i-1}{n}a\right)^2$

$=\frac{(i-1)^2}{n^2}a^2$

$M_i = l.u.b.\ \{x^2 : x_{i-1} \le x \le x_i\}$

$= l.u.b.\left\{x^2 : \frac{i-1}{n}a \le x \le \frac{i}{n}a\right\}$

$= \left(\frac{i}{n}a\right)^2$

$= \frac{i^2 a^2}{n^2}$

Also $\delta_i = x_i - x_{i-1} = \frac{i}{n}a - \frac{i-1}{n}a = \frac{a}{n}$, for $i = 1, 2, \ldots, n$

$L(P, f) = \sum_{i=1}^{n} m_i \delta_i$

$= \sum_{i=1}^{n} \frac{(i-1)^2 a^2}{n^2} \cdot \frac{a}{n}$

$= \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2$

$= \frac{a^3}{n^3}\left[0^2 + 1^2 + 2^2 + \ldots + (n-1)^2\right]$

$= \frac{a^3}{n^3}\left[\frac{(n-1)\{(n-1)+1\}\{2(n-1)+1\}}{6}\right]$

$\left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{a^3 (n-1)(2n-1)}{6n^2}$

$U(P, f) = \sum_{i=1}^{n} M_i \delta_i$

$= \sum_{i=1}^{n} \frac{i^2 a^2}{n^2} \cdot \frac{a}{n}$

$= \frac{a^3}{n^3} \sum_{i=1}^{n} i^2$

$= \frac{a^3}{n^3}\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right)$

$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{a^3 (n+1)(2n+1)}{6n^2}$

$\therefore \underline{\int_0^a} f(x)\,dx = \underset{\|P\| \to 0}{Lt}\ L(P, f)$

$= \underset{\frac{a}{n} \to 0}{Lt} \frac{a^3 (n-1)(2n-1)}{6n^2}$

$= \underset{n \to \infty}{Lt}\left[\frac{a^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$

$= \frac{a^3}{6}(1-0)(2-0)$

$= \frac{a^3}{3}$

$\therefore \overline{\int_0^a} f(x)\,dx = \underset{\|P\| \to 0}{Lt}\ U(P, f)$

$= \underset{\frac{a}{n} \to 0}{Lt} \cdot \frac{a^3 (n+1)(2n+1)}{6n^2}$

$= \underset{n \to \infty}{Lt}\left[\frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right]$

$= \frac{a^3}{6}(1+0)(2+0)$

$= \frac{a^3}{3}$

$\therefore \underline{\int_0^a} f(x)\,dx = \overline{\int_0^a} f(x)\,dx = \frac{a^3}{3}$

$\therefore f \in R[0, a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$

2

IMPROPER INTEGRAL

1. State Dirichlet's test for improper integrals and hence show $\int_0^\infty \sin x^2 dx$ is convergent. **(September 2013)**

Sol. Statement:–

If g is bounded and monotonic and tends to zero as x $\to\infty$ and $\int_a^t f(x)dx$ is bounded for $t \geq a$ then $\int_a^\infty f(x)g(x)dx$ is convergent at ∞

$\sin(x^2) = \frac{1}{2x}(2x)\sin(x^2)$

$\therefore \int_0^\infty \sin(x^2)dx = \int_0^1 \sin(x^2)dx + \int_1^\infty \sin(x^2)dx$

But $\int_0^1 \sin(x^2)dx$ is a proper integral

$\therefore$ we discuss the convergence of $\int_1^\infty \sin(x^2)dx$ at ∞

Now $\int_1^\infty \sin(x^2)dx = \int_1^\infty (2x\sin(x^2))\frac{1}{2x}dx$

Take $f(x) = 2x\sin(x^2)$ and $g(x) = \frac{1}{2x}$

$\therefore \int_1^t f(x)dx = \int_1^t 2x\sin(x^2)dx$

Put $x^2 = y \Rightarrow 2x\,dy = dy$

$= \int_1^{t^2} \sin y\,dy = (-\cos y)_1^{t^2}$

$= \cos 1 - \cos t^2 \quad \forall t \geq 1$

$\therefore \left|\int_1^t f(x)dx\right| = |\cos 1 - \cos t^2| \leq |\cos 1| + |\cos t^2|$

$\leq 1 + 1 = 2, \quad \forall t \geq 1$ $\quad [\because |\cos\theta| \leq 1]$

and $g(x) = \frac{1}{2x}$ is monotonic decreasing and $\to 0$, as x $\to\infty$

$\therefore$ by Dirichlet's test, $\int_1^\infty 2x\sin(x^2)\frac{1}{2x}dx$ is convergent

or $\int_1^\infty \sin(x^2)dx$ is convergent at ∞ and consequently

$\int_0^\infty \sin(x^2)dx$ is convergent at ∞.

2. State and prove Abel's test for convergence of improper integral. **(September 2013)**

Sol. Since g is monotonic in (a, ∞)

$\therefore$ it is integrable in $(a, t)\ \forall t \geq a$

Also f is integrable in (a, t)

$\therefore$ by second mean value theorem

$\int_{t_1}^{t_2} fg\,dx = g(t_1)\int_{t_1}^{\xi} f(x)dx + g(t_2)\int_{\xi}^{t_2} f(x)dx$,

for $a < t_1 \leq \xi \leq t_2$ (1)

Let $\in > 0$ be given

$\because$ g is bounded in (a, ∞)

$\therefore \exists$ a positive number k s.t.

$|g(x)| \leq k \quad \forall x \geq a$. In particular

$|g(t_1)| \leq k$ and $|g(t_2)| \leq k$ (2)

Since $\int_0^\infty f(x)dx$ is convergent

$\therefore$ by Cauchy's test, a number M exists such that

$$\left|\int_{t_1}^{t_2} f dx\right| < \frac{\in}{2M} \; \forall t_1, t_2 \geq M \qquad (3)$$

Let t_1, t_2 of (1) be $\geq$ M, so that ξ lying between t_1, t_2 is also $\geq$ M.

$\therefore$ from (3)

$$\left|\int_{t_1}^{\xi} f dx\right| < \frac{\in}{2k}, \left|\int_{\xi}^{t_2} f dx\right| < \frac{\in}{2k} \qquad (4)$$

i.e. from (1), (2) and (4) it follows that M exists such that for $t_1, t_2 \geq M$

$$\left|\int_{t_1}^{t_2} fg dx\right| \leq |g(t_1)|\left|\int_{t_1}^{\xi} f dx\right| + |g(t_2)|\left|\int_{\xi}^{t_2} f dx\right|$$

$$< k.\frac{\in}{2k} + k.\frac{\in}{2k} = \in$$

$\therefore$ By Cauchy's test, $\int_a^{\infty} fg dx$ is convergent at ∞.

3. **Prove that improper integral $\int_a^{\infty} \frac{dx}{x^n} (a > 0)$ is convergent iff n > 1. Hence examine the convergence of $\int_1^{\infty} \frac{x^3}{(1+x)^5} dx$.** **(April 2013)**

Sol. $I = \int_a^{\infty} \frac{dx}{x^n}$

Consider $\int_a^{t} \frac{dx}{x^n} = \begin{cases} [\log x]_a^t = \log \frac{t}{a}, & \text{when } n = 1 \\ \left[\frac{x^{-n+1}}{-n+1}\right]_a^t = \frac{1}{(1-n)}\left[t^{1-n} - a^{1-n}\right], & \text{when } n \neq 1 \end{cases}$

When $n = 1, \lim_{t\to\infty} \int_a^t \frac{dx}{x^n} = \lim_{t\to\infty} \log \frac{t}{a} \to \infty$

$\therefore$ I diverges at ∞

When $n < 1, \lim_{t\to\infty} \int_a^t \frac{dx}{x^n} = \lim_{t\to\infty} \frac{1}{1-n}\left[t^{1-n} - a^{1-n}\right] \to \infty$

$\therefore$ I diverges at ∞

When $n > 1, \lim_{t\to\infty} \int_a^t \frac{dx}{x^n} = \lim_{t\to\infty} \frac{1}{1-n}\left[\frac{1}{t^{n-1}} - \frac{1}{a^{n-1}}\right] = \frac{1}{(n-1)}\left(\frac{1}{a^{n-1}}\right) = \text{finite}$

$\therefore$ I converges at ∞

Hence $\int_a^{\infty} \frac{dx}{x^n}$ $(a > 0)$ is convergent, iff n > 1

Now $\int_1^{\infty} \frac{x^3}{(1+x)^5} dx$, $f(x) = \frac{x^3}{(1+x)^5}$ Take $\phi(x) = \frac{1}{x^2}$

$$\therefore \frac{f(x)}{\phi(x)} = \frac{\frac{x^3}{(1+x)^5}}{\frac{1}{x^2}} = \frac{x^5}{(1+x)^5} = \frac{1}{\left(1+\frac{1}{x}\right)^5}$$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 1 \neq 0, \infty$$

$\therefore$ By practical comparison test $\int_1^{\infty} f(x)dx$ and $\int_1^{\infty} \phi(x)dx$ behave alike

But $\int_1^{\infty} \frac{dx}{x^2}$ is convergent $\Rightarrow \int_1^{\infty} \frac{x^3}{(1+x)^5} dx$ is convergent.

4. **Examine the convergence of improper integral $\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx \, (m > 0)$.** **(April 2013)**

Sol. $I = \int_0^{\frac{\pi}{2}} \frac{\sin^m x}{x^n} dx$

and $f(x) = \frac{\sin^m x}{x^n} = \left(\frac{\sin x}{x}\right)^m \cdot \frac{1}{x^{n-m}}$

Case (1) When $n - m < 0$ *i.e.* $m - n > 0$

$$\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}$$

$$= \underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^m \cdot x^{m-n} = 1.0 = 0 = \text{finite}$$

$\therefore$ I is proper integral when $n - m < 0$

Case 2. When $n - m = 0$

$$\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}} = \underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^m \frac{1}{x^0} = 1.1 = 1 = \text{finite}$$

$\therefore$ I is a proper integral when $n - m = 0$

Case 3. When $n - m > 0$

$$f(x)=\left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}$$

Take $g(x)=\frac{1}{x^{n-m}}$

$$\therefore \ \underset{x\to 0}{Lt}\frac{f(x)}{g(x)}=\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}\times x^{n-m}$$

$$=\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^m=1=\text{finite, which is non zero non infinity}$$

$\therefore$ By practical comparison test $\int_0^{\frac{\pi}{2}} f(x)dx$ and $\int_0^{\frac{\pi}{2}} g(x)dx$ behave alike.

But $\int_0^{\frac{\pi}{2}} g(x)dx=\int_0^{\frac{\pi}{2}}\frac{dx}{x^{n-m}}$ is convergent if $n-m<1$ (By p test)

$\therefore \int_0^{\frac{\pi}{2}} f(x)dx=\int_0^{\frac{\pi}{2}}\left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}dx$ is convergent if $n-m<1$

By combining above three cases we get

$I=\int_0^{\frac{\pi}{2}}\frac{\sin^m x}{x^n}dx$ is convergent if $n-m<1$ i.e. $n<m+1$

5. Test for convergence of the Integral $\int_{-\infty}^{+\infty}\frac{e^x}{1+e^{2x}}dx$. Also find the value of Integral if it is convergent. (September 2012)

Sol. $\int_{-\infty}^{\infty}\frac{e^x}{1+e^{2x}}dx=\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}=\int_{-\infty}^{0}\frac{dx}{e^x+e^{-x}}+\int_0^{\infty}\frac{dx}{e^x+e^{-x}}$

$$=2\int_0^{\infty}\frac{dx}{e^x+e^{-x}}=2\int_0^{\infty}\frac{e^x}{e^{2x}+1}dx$$

$$=2\lim_{t\to\infty}\int_0^t\frac{e^x}{e^{2x}+1}dx=2\lim_{t\to\infty}\left[\tan^{-1}(e^x)\right]_0^t$$

$$=2\lim_{t\to\infty}\left[\tan^{-1}(e^t)-\tan^{-1}1\right]$$

$$=2\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=2\left(\frac{\pi}{4}\right)=\frac{\pi}{2}$$

$\therefore$ the given integral converges.

6. State Dirichlet's test for convergence of improper integral. Use it to discuss the convergence of $\int_1^{\infty}\frac{\sin x}{x^p}dx$. (April 2012)

Sol. Statement: If g is bounded and monotonic and tends to zero as $x\to\infty$ and $\int_a^t f(x)dx$ is bounded for $t\ge a$ then $\int_a^{\infty} f(x)g(x)dx$ is convergent at ∞

Let $f(x)=\sin x$ and $g(x)=\frac{1}{x^p}(p>0)$

$$\left|\int_1^t \sin x dx\right|=\left|\left[-\cos x\right]_1^t\right|=\left|\cos 1-\cos t\right|\le\left|\cos 1\right|+\left|\cos t\right|$$

$$\le 1+1=2 \qquad \forall t\ge 1 \qquad \left[\because \ |\cos\theta|\le 1\right]$$

$\therefore \int_1^t \sin x dx$ is bounded

and $g(x)$ is monotonic decreasing $\to 0$, as $x\to\infty$

$\therefore$ by Dirichlet's test $\int_1^{\infty}\frac{\sin x}{x^p}dx$ is convergent.

7. Prove that $\int_1^{\infty}\frac{\sin x}{x}dx$ is convergent but not absolutely convergent. (April 2012)

Sol. The given integral is $\int_1^{\infty}\frac{\sin x}{x}dx$

Let $t>1$

$$\int_1^t\frac{\sin x}{x}dx=\left[\frac{-1}{x}\cos x\right]_1^t-\int_1^t-\cos x\left(\frac{-1}{x^2}\right)dx$$

$$=\left(\frac{-\cos t}{t}\right)-\left(\frac{-\cos 1}{1}\right)-\int_1^t\frac{\cos x}{x^2}dx$$

$$=-\frac{\cos t}{t}+\cos 1-\int_1^t\frac{\cos x}{x^2}dx \qquad (1)$$

Now $\left|\frac{\cos x}{x^2}\right| = \frac{|\cos x|}{x^2} \le \frac{1}{x^2}$ $\qquad [\because |\cos x| \le 1]$

Since $\int_1^\infty \frac{dx}{x^2}$ converges

$\therefore \int_1^\infty \frac{\cos x}{x^2}dx$ converges absolutely and hence converges

From (1),

$$\int_1^\infty \frac{\sin x}{x}dx = \lim_{t\to\infty}\int_0^t \frac{\sin x}{x}dx = \text{Lim}_{t\to\infty}\left[\frac{-\cos t}{t} + \cos 1 - \int_1^t \frac{\cos x}{x^2}dx\right]$$

$$= Lim_{t\to\infty}\left[0 + \cos 1 - \int_1^t \frac{\cos x}{x^2}dx\right] = \cos 1 - \int_1^\infty \frac{\cos x}{x^2}dx$$

Since integral on R.H.S. is cgt.

$\therefore \int_1^\infty \frac{\sin x}{x}dx$ is also convergent.

Now to show that $\int_1^\infty \frac{\sin x}{x}dx$ does not converge absolutely.

For $t = n\pi$, $n \in \mathbb{N}$, we have

$$\int_\pi^{n\pi}\left|\frac{\sin x}{x}\right|dx = \int_\pi^{n\pi}\frac{|\sin x|}{x}dx = \sum_{m=1}^{n-1}\int_{m\pi}^{(m+1)\pi}\frac{|\sin x|}{x}dx$$

$$\ge \sum_{m=1}^{n-1}\frac{1}{(m+1)\pi}\int_{m\pi}^{(m+1)\pi}|\sin x|dx$$

$$\left[\because m\pi \le x \le (m+1)\pi \Rightarrow \frac{1}{x} \ge \frac{1}{(m+1)\pi}\right]$$

$$\ge \frac{1}{\pi}\sum_{m=1}^{n-1}\frac{1}{(m+1)}\int_0^\pi |\sin(u+m\pi)|du \qquad \text{[By putting x = u + m}\pi\text{]}$$

Now $\sin(u+m\pi) = \sin u\cos m\pi + \cos u \sin m\pi = \sin u \cos m\pi$

Since $\cos m\pi = \pm 1$ $\qquad \therefore |\sin(u+m\pi)| = |\sin u|$

Hence if $0 \le u \le \pi$, then $|\sin(u+m\pi)| = \sin u$

Thus $\int_\pi^{n\pi}\left|\frac{\sin x}{x}\right|dx \ge \frac{1}{\pi}\sum_{m=1}^{n-1}\frac{1}{m+1}\int_0^\pi \sin u\,du = \frac{1}{\pi}\sum_{m=1}^{n-1}\frac{1}{m+1}2\int_0^{\pi/2}\sin u\,du$

$$= \frac{2}{\pi}\sum_{m=1}^{n-1}\frac{1}{m+1} \qquad \left[\because \int_0^{\frac{\pi}{2}}\sin u\,du = 1\right]$$

$$= \frac{2}{\pi}\sum_{k=2}^{n}\frac{1}{k} \qquad \text{[By putting m + 1 = k]}$$

$$\Rightarrow \int_\pi^{n\pi}\left|\frac{\sin x}{x}\right|dx \ge \frac{2}{\pi}\sum_{k=2}^{n}\frac{1}{k}$$

Now $\sum_{n=1}^\infty \frac{1}{n}$ is divergent

$\therefore \text{Lim}_{t\to\infty}\int_\pi^t \left|\frac{\sin x}{x}\right|dx$ is also divergent

$\Rightarrow \int_1^\infty \left|\frac{\sin x}{x}\right|dx = \int_1^\pi \left|\frac{\sin x}{x}\right|dx + \int_\pi^\infty \left|\frac{\sin x}{x}\right|dx$ is divergent

Hence $\int_1^\infty \frac{\sin x}{x}dx$ is convergent but not absolutely convergent

8. Test the convergence of improper integral $\int_a^\infty \frac{\sin x}{x^2}dx$ where a > 0.

(September 2011)

Sol. Given $\int_a^\infty \frac{\sin x}{x^2}dx$, $\qquad a > 0$

Now $\left|\frac{\sin x}{x^2}\right| = \frac{|\sin x|}{|x^2|} = \frac{|\sin x|}{x^2} \le \frac{1}{x^2}$, $\qquad \forall x \ge 1 \qquad [\because |\sin x| \le 1]$

Since $\int_a^\infty \frac{dx}{x^2}$ converges using p – test

$\therefore$ By comparison test, $\int_a^\infty \left|\frac{\sin x}{x^2}\right|dx$ converges

$\Rightarrow \int_a^\infty \frac{\sin x}{x^2}dx$ converges absolutely

$\Rightarrow \int_a^{\infty} \frac{\sin x}{x^2}$ is convergent

9. Show improper integral $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$ is convergent and find it if given $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

(April 2011)

Sol. $\because \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \int_0^{1} \frac{\sin^2 x}{x^2} dx + \int_1^{\infty} \frac{\sin^2 x}{x^2} dx$

x = 0, is not a point of infinite discontinuity

$\therefore \int_0^{1} \frac{\sin^2 x}{x^2} dx$ is a proper integral

Consider $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$

We know $\sin^2 x \le 1, \quad \forall x > 1$

$\Rightarrow \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$

But $\int_1^{\infty} \frac{dx}{x^2}$ is convergent at ∞

$\therefore \int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ is also convergent at ∞

Hence, $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$ is convergent

Further, $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \int_0^{\infty} (\sin^2 x)\left(\frac{1}{x^2}\right) dx$

$= \left[(\sin^2 x)\left(-\frac{1}{x}\right)\right]_0^{\infty} + \int_0^{\infty} \frac{2 \sin x \cos x}{x} dx$

$= 0 + \int_0^{\infty} \frac{\sin 2x}{x} dx$

$= \int_0^{\infty} \frac{\sin 2x}{x} dx$ Put 2x = y

$\therefore$ 2 dx = dy $\Rightarrow$ dx $= \frac{1}{2} dy$

$= \int_0^{\infty} \frac{2 \sin y}{y} \times \frac{1}{2} dy$

$= \int_0^{\infty} \frac{\sin y}{y} dy = \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ (given)

$\therefore \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

10. Discuss convergence of improper integral $\int_0^{1} \frac{\log x}{\sqrt{x}} dx$.

(April 2010)

Sol. $I = \int_0^{1} \frac{\log x}{\sqrt{x}} dx$

$\because \frac{\log x}{\sqrt{x}}$ is negative in [0, 1]

$\therefore$ take $f(x) = -\frac{\log x}{\sqrt{x}}$

$= \frac{\log\left(\frac{1}{x}\right)}{\sqrt{x}}$

$\therefore$ '0' is the only point of infinite discontinuity of f

Take $g(x) = \frac{1}{x^{\frac{3}{4}}}$

$\therefore \frac{f(x)}{g(x)} = x^{\frac{3}{4}} \frac{\log\left(\frac{1}{x}\right)}{\sqrt{x}} = x^{\frac{1}{4}} \log \frac{1}{x}$

$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} x^{\frac{1}{4}} \log \frac{1}{x} = 0$

But $\int_0^{1} g(x) dx = \int_0^{1} \frac{1}{x^{\frac{3}{4}}} dx$ converges $\left(\because p = \frac{3}{4} < 1\right)$

$\therefore \int_0^{1} \frac{\log x}{\sqrt{x}} dx$ converges at 0.

11. Examine the convergence of $\int_1^{2} \frac{dx}{2x - x^2}$. **(September 2009)**

Sol. Let $I = \int_1^{2} \frac{dx}{2x - x^2} = \int_1^{2} \frac{dx}{x(2-x)}$

2 is the only point of infinite discontinuity of f

$$\therefore\ I = \lim_{\varepsilon\to 0^+}\int_1^{2-\varepsilon}\frac{dx}{(1)^2-(x-1)^2}$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{2.1}\log\left[\frac{1+(x-1)}{1-(x-1)}\right]_1^{2-\varepsilon}$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{2}\log\left[\frac{x}{2-x}\right]_1^{2-\varepsilon}$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{2}\log\left(\frac{2-\varepsilon}{\varepsilon}\right) \text{ which doesn't exist}$$

$\therefore$ I diverges

12. Examine the convergence of $\int_0^2\frac{dx}{x^2-4x+3}$. **(April 2009)**

Sol. $$I = \int_0^2\frac{dx}{x^2-4x+3} = \int_0^2\frac{dx}{(x-1)(x-3)}$$

$$= \int_0^1\frac{dx}{(x-1)(x-3)} + \int_1^2\frac{dx}{(x-1)(x-3)}$$

$$= \lim_{\varepsilon\to 0+}\int_0^{1-\varepsilon}\frac{dx}{(x-2)^2-(1)^2} + \lim_{\varepsilon'\to 0+}\int_{1+\varepsilon'}^2\frac{dx}{(x-2)^2-1}$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{2}\left[\log\frac{x-2-1}{x-2+1}\right]_0^{1-\varepsilon} + \lim_{\varepsilon'\to 0+}\frac{1}{2}\left\{\log\left|\frac{x-3}{x-1}\right|\right\}_{1+\varepsilon'}^2$$

$$= \frac{1}{2}\lim_{\varepsilon\to 0+}\left\{log\left|\left(\frac{1-\varepsilon-3}{-\varepsilon}\right)\right| - log 3\right\} + \frac{1}{2}\lim_{\varepsilon'\to 0+}\left(log 1 - log\left|\frac{\varepsilon'-2}{\varepsilon'}\right|\right)$$

$$= \frac{1}{2}\lim_{\varepsilon\to 0+}\log\frac{2+\varepsilon}{3\varepsilon} + \frac{1}{2}\lim_{\varepsilon'\to 0+}\log\left|\frac{\varepsilon'}{2-\varepsilon'}\right|$$

Which does not exist

$\therefore$ I diverges.

3

BETA AND GAMMA FUNCTIONS

1. Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}; m>0,\ n>0$. **(September 2013)**

Sol. We have

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$$

Put $x = tz$, $\therefore\ dx = tdz$

Now $x = 0$ $\Rightarrow z = 0$

and $x\to\infty$ $\Rightarrow z\to\infty$

$$\therefore\ \Gamma(n) = \int_0^\infty (tz)^{n-1}e^{-tz}.tdz$$

$$\Rightarrow \Gamma(n) = \int_0^\infty t^n z^{n-1}e^{-tz}dz$$

Multiplying both sides by $e^{-t}t^{m-1}$, we get,

$$e^{-t}t^{m-1}\Gamma(n) = e^{-t}t^{m-1}\int_0^\infty t^n z^{n-1}e^{-tz}dz$$

$$\text{or } \Gamma(n).e^{-t}t^{m-1} = \int_0^\infty t^{m+n-1}z^{n-1}e^{-t(z+1)}dz$$

Integrating both sides w.r.t. t between the limits 0 to ∞, we get,

$$\Gamma(n)\int_0^\infty e^{-t}t^{m-1}dt = \int_0^\infty\left[\int_0^\infty t^{m+n-1}z^{n-1}e^{-t(z+1)}dz\right]dt$$

$$\Rightarrow \Gamma(n)\int_0^\infty e^{-t}t^{m-1}dt = \int_0^\infty z^{n-1}\left[\int_0^\infty t^{m+n-1}e^{-t(z+1)}dt\right]dz \qquad (1)$$

Put $t(z+1)=y \quad \therefore t=\dfrac{y}{z+1} \quad \Rightarrow dt=\dfrac{dy}{z+1}$

Now $t=0 \quad \Rightarrow y=0$

and $t\to\infty \quad \Rightarrow y\to\infty$

$\therefore$ from (1), we get

$$\Gamma(n)\int_0^\infty e^{-t}t^{m-1}dt=\int_0^\infty z^{n-1}\left[\int_0^\infty\left(\frac{y}{z+1}\right)^{m+n-1}e^{-y}\frac{dy}{z+1}\right]dz$$

$$\therefore\ \Gamma(n)\Gamma(m)=\int_0^\infty\frac{z^{n-1}}{(z+1)^{m+n}}\left[\int_0^\infty y^{m+n-1}e^{-y}dy\right]dz$$

$$\Rightarrow \Gamma(m)\Gamma(n)=\int_0^\infty\frac{z^{n-1}}{(z+1)^{m+n}}\Gamma(m+n)dz$$

$$\Rightarrow \Gamma(m)\Gamma(n)=\Gamma(m+n)\int_0^\infty\frac{z^{n-1}}{(z+1)^{m+n}}dz$$

$$\Rightarrow \Gamma(m)\Gamma(n)=\Gamma(m+n)\beta(m,n)$$

$$\Rightarrow \beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

2. Prove that $\int_0^1\dfrac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx=\dfrac{1}{a^n(a+b)^m}\beta(m,n)$ where m, n > 0.

(April 2013, September 2012)

Sol. Let $I=\displaystyle\int_0^1\frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx$

Put $\dfrac{x}{a+bx}=\dfrac{t}{a+b}, \quad \therefore ax+bx=at+btx$

$\Rightarrow x(a+b-bt)=at, \quad \Rightarrow \quad x=\dfrac{at}{a+b-bt}$

$$\therefore dx=\left[\frac{(a+b-bt).a-at(-b)}{(a+b-bt)^2}\right]dt$$

$$=\frac{a(a+b)}{(a+b-bt)^2}dt$$

Now $x=0\Rightarrow t=0$ and $x=1\Rightarrow t=1$

$$\therefore\ I=\int_0^1\frac{\left(\dfrac{at}{a+b-bt}\right)^{m-1}\left(1-\dfrac{at}{a+b-bt}\right)^{n-1}}{\left[a+b\left(\dfrac{at}{a+b-bt}\right)\right]^{m+n}}\frac{a(a+b)}{(a+b-bt)^2}dt$$

$$=\int_0^1\frac{\dfrac{a^{m-1}.t^{m-1}}{(a+b-bt)^{m-1}}\cdot\dfrac{(a+b)^{n-1}(1-t)^{n-1}}{(a+b-bt)^{n-1}}\cdot\dfrac{a(a+b)}{(a+b-bt)^2}}{\dfrac{a^{m+n}(a+b)^{m+n}}{(a+b-bt)^{m+n}}}dt$$

$$=\int_0^1\frac{(a+b)^{n-1+1}a^{m-1+1}}{a^{m+n}(a+b)^{m+n}}t^{m-1}(1-t)^{n-1}dt$$

$$=\frac{1}{a^n(a+b)^m}\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

$$=\frac{1}{a^n(a+b)^m}\beta(m,n)$$

3. Use the relation $\left\{\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\text{ where } m,n>0\right\}$ between Beta and Gamma function. Prove that: (i) $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$ (ii) $\displaystyle\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

(April 2012)

Sol. **(i)** We have

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Put $m=\dfrac{1}{2},\quad n=\dfrac{1}{2}$

$$\therefore\ \beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right)=\left[\Gamma\left(\frac{1}{2}\right)\right]^2 \qquad [\because \Gamma(1)=1]$$

$$\Rightarrow \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx \qquad (1)$$

Put $x = \sin^2\theta$, $\Rightarrow dx = 2\sin\theta\cos\theta d\theta$

When x = 0, $\theta = 0$

When x = 1, $\theta = \frac{\pi}{2}$

$\therefore$ from (1), we get,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}}.2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\cos\theta}.2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1\, d\theta = 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2}-0\right] = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(ii)Let $I = \int_0^\infty e^{-x^2}dx$

Put $x^2 = t$, $\therefore 2xdx = dt \Rightarrow dx = \frac{1}{2x}dt = \frac{1}{2\sqrt{t}}dt$

Now x = 0, t = 0 and $x \to \infty \Rightarrow t \to \infty$

$$\therefore I = \int_0^\infty e^{-t}.\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\int t^{-\frac{1}{2}}e^{-t}dt$$

$$= \frac{1}{2}\Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

4. Prove that $\int_0^1 \sqrt{\frac{1-x}{x}}dx = \frac{\pi}{2}$. **(April 2012)**

Sol. $\int_0^1 \sqrt{\frac{1-x}{x}}dx = \int_0^1 x^{-1/2}(1-x)^{1/2}dx$

$$= \int_0^1 x^{\frac{1}{2}-1}(1-x)^{3/2-1}dx$$

$$= \beta\left(\frac{1}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right).\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$= \frac{\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{1} = \frac{\pi}{2}$$

5. Define Beta function $\beta(m,n)$ and show $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$

Hence show $\int_0^{\pi/2} \sin^7\theta d\theta = \frac{16}{35}$. **(September 2011)**

Sol. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx;\quad m,n > 0$

Put $x = \sin^2\theta$ $\therefore dx = 2\sin\theta\cos\theta d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \beta(m,n) = \int_0^{\pi/2} \left(\sin^2\theta\right)^{m-1}\left(1-\sin^2\theta\right)^{n-1} 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-2+1} (\cos\theta)^{2n-2+1} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Now $\int_0^{\pi/2} \sin^7\theta \, d\theta = \int_0^{\pi/2} \sin^{2(4)-1}\theta \cos^{2\left(\frac{1}{2}\right)-1}\theta \, d\theta$

$$= \beta\left(4, \frac{1}{2}\right)$$

$$= \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} \qquad \left[\because \beta(m,n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}\right]$$

$$= \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{3!\sqrt{\pi}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} = \frac{6\sqrt{\pi}}{\frac{105}{8}\sqrt{\pi}} = \frac{16}{35}$$

6. State Gamma function and use it to show $\int_0^\infty e^{-x^3} dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$. **(April 2011)**

Sol. Gamma function

The second Eulerian integral $\int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$ is called a Gamma function and is denoted by $\Gamma(n)$.
The quantity n is positive but not necessarily integer.

Let $I = \int_0^\infty e^{-x^3} dx$

Put $x^3 = t$, $\quad \therefore 3x^2 dx = dt$

$\therefore dx = \frac{1}{3x^2} dt = \frac{1}{3t^{\frac{2}{3}}} dt$

Now x = 0 $\Rightarrow$ t = 0 and x $\to\infty \Rightarrow$ t $\to\infty$

$$= \int_{t=0}^{\infty} e^{-t} \frac{dt}{3(t)^{\frac{2}{3}}} = \frac{1}{3}\int_0^\infty t^{-\frac{2}{3}} e^{-t} dt$$

$$= \frac{1}{3}\Gamma\left(-\frac{2}{3}+1\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

7. Define Beta and Gamma functions. Using relation between these functions show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. **(September 2010)**

Sol. Beta function

The first Eulerian integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$

Where m > 0, n > 0 is called a Beta Function and is denoted by B(m, n).
The quantities m, n are positive but not necessarily integers.

Gamma Function

The second Eulerian integral $\int_0^\infty e^{-x} x^{n-1} dx, n > 0$ is called a Gamma function and is denoted by $\Gamma(n)$.
The quantity n is positive but not necessarily integer.
We have

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Put $m = \frac{1}{2}$, $n = \frac{1}{2}$

$$\therefore \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \qquad [\because \Gamma(1) = 1]$$

$$\Rightarrow \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1} dx$$

$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx$$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx \qquad (1)$$

Put $x = \sin^2\theta$, $\Rightarrow$ $dx = 2\sin\theta\cos\theta d\theta$

When x = 0, $\theta = 0$

When x = 1, $\theta = \frac{\pi}{2}$

$\therefore$ from (1), we get,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}}.2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\cos\theta}.2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1\, d\theta = 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2} - 0\right] = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

8. **Show** $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}dx \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}} = \pi$. **(April 2010)**

Sol. L.H.S. = $\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{2}} dx \int_0^{\frac{\pi}{2}} \frac{1}{(\sin x)^{\frac{1}{2}}}dx$

$$= \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{1}{2}}(\cos x)^0 dx \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{-1}{2}}(\cos x)^0 dx$$

$$= \left(\frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}\right)\left(\frac{1}{2}\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\frac{\sqrt{\pi}\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$= \pi =$ R.H.S.

9. **Evaluate** $\int_0^{\infty} \frac{x^2}{1+x^4}dx$. **(September 2009)**

Sol. Let $I = \int_0^{\infty} \frac{x^2}{1+x^4}dx$

Put $x^2 = \tan t$ or $x = \sqrt{\tan t}$

$\therefore dx = \frac{1}{2}(\tan t)^{-1/2}\sec^2 t dt$

Now x = 0 $\Rightarrow$ t = 0

x = ∞ $\Rightarrow t = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \frac{\tan t}{1+\tan^2 t}\frac{1}{2}(\tan t)^{-1/2}\sec^2 t dt$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\tan t}dt$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{(\sin t)^{1/2}}{(\cos t)^{1/2}}dt = \frac{1}{2}\int_0^{\pi/2} (\sin t)^{1/2}(\cos t)^{-1/2} dt$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{-\frac{1}{2}+1}{2}\right)}\right]$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

But $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$

[By duplication formula]

$$\Rightarrow I = \frac{\sqrt{2}\pi}{4} = \frac{\pi}{2\sqrt{2}}$$

10. Prove that: $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\frac{\pi}{2^{2m-1}}}\Gamma(2m)$ **where m is +ve.**

(April 2009)

Sol. We have

$$\beta(m,m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx$$

Put $x = \sin^2\theta$, $\therefore dx = 2\sin\theta\cos\theta d\theta$

When $x = 0, \quad \theta = 0$

When $x = 1, \quad \theta = \frac{\pi}{2}$

$$\therefore \beta(m,m) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{m-1}.2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta)^{m-1}.2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2m-2}\theta.\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2m-1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta = 2\int_0^{\frac{\pi}{2}} \left(\frac{(\sin 2\theta)}{2}\right)^{2m-1} d\theta$$

$$\therefore \beta(m,m) = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta$$

Put $2\theta = z$ i.e. $\theta = \frac{1}{2}z$

$$\Rightarrow d\theta = \frac{1}{2}dz$$

When $\theta = 0, \quad z = 0$

When $\theta = \frac{\pi}{2}, \quad z = \pi$

$$\therefore \beta(m,m) = \frac{2}{2^{2m-1}}\int_0^{\pi} (\sin^{2m-1} z).\frac{1}{2}dz$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1} z dz$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} z\, dz$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} \sin^{2m-1} z\cos^0 z dz$$

$$= \frac{2}{2^{2m-1}}.\frac{1}{2}\frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2m-1+1}{2}+\frac{0+1}{2}\right)}$$

$$\therefore \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}.\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}\right)}$$

$$\Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}.\frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\Rightarrow \Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

4

INTEGRALS AS FUNCTIONS OF A PARAMETER

1. If $|a|<1$, then evaluate $\int_0^\pi \frac{\log(1+a\cos x)}{\cos x}dx$, where "a" is parameter.

(September 2013, September 2009)

Sol. Here 'a' is parameter

Let $\phi(a)=\int_0^\pi\left[\frac{\log(1+a\cos x)}{\cos x}\right]dx$ (A)

It can be easily established that $x=\frac{\pi}{2}$ is the only point of removable discontinuity.

Differentiating w.r.t. a by applying Leibnitz's Rule

$$\frac{d}{da}(\phi(a))=\int_0^\pi\frac{\partial}{\partial a}\left[\frac{\log(1+a\cos x)}{\cos x}\right]dx$$

$$=\int_0^\pi\frac{1}{\cos x}\frac{\partial}{\partial a}\log(1+a\cos x)dx$$

$$=\int_0^\pi\frac{1}{\cos x}\frac{1}{1+a\cos x}\cos x dx$$

$$=\int_0^\pi\frac{1}{1+a\cos x}dx$$

$$=\int_0^\pi\frac{1}{1+a\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}dx$$

$$=\int_0^\pi\frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}+a-a\tan^2\frac{x}{2}}dx \qquad (\because |a|<1)$$

$$=\int_0^\pi\frac{\sec^2\frac{x}{2}}{(1+a)+(1-a)\tan^2\frac{x}{2}}dx$$

Let $\tan\frac{x}{2}=t$

$\therefore \sec^2\frac{x}{2}dx=2dt$

$x=0 \Rightarrow t=0$

$x=\pi \Rightarrow t=\infty$

$$=\int_0^\infty\frac{2}{(1+a)+(1-a)t^2}dt$$

$$=\frac{2}{1-a}\int_0^\infty\frac{1}{\left(\frac{1+a}{1-a}\right)+t^2}dt$$

$$=\frac{2}{1-a}\cdot\sqrt{\frac{1-a}{1+a}}\tan^{-1}\frac{t\sqrt{1-a}}{\sqrt{1+a}}\Bigg|_0^\infty$$

$$=\frac{2}{\sqrt{1-a^2}}\{\tan^{-1}\infty-\tan^{-1}0\}$$

$$=\frac{2}{\sqrt{1-a^2}}\times\frac{\pi}{2}=\frac{\pi}{\sqrt{1-a^2}}$$

Integrating w.r.t. 'a'

$\phi(a)=\pi\sin^{-1}a+c$ (B)

is required value of given integral where c is constant of integration

Put a = 0 in (A) and (B)

$(A)\Rightarrow\phi(0)=\int_0^\pi\frac{\log 1}{\cos x}dx=0$

$(B)\Rightarrow\phi(0)=\pi\sin^{-1}(0)+c$

= c

$\therefore$ c = 0

$\therefore \int_0^\pi\frac{\log(1+a\cos x)}{\cos x}dx=\pi\sin^{-1}a$

2. Find the value of $\int_0^\pi \frac{dx}{a+b\cos x}$ (where $a>0,\ |b|<a$) and hence deduce that

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \text{ and } \int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2} = \frac{-\pi b}{(a^2-b^2)^{3/2}}.$$

(April 2013)

Sol. Let $I = \int_0^\pi \frac{dx}{a+b\cos x}$

$$= \int_0^\pi \frac{dx}{a\cos^2\left(\frac{x}{2}\right)+a\sin^2\left(\frac{x}{2}\right)+b\cos^2\left(\frac{x}{2}\right)-b\sin^2\left(\frac{x}{2}\right)}$$

$$= \int_0^\pi \frac{dx}{(a+b)\cos^2\left(\frac{x}{2}\right)+(a-b)\sin^2\left(\frac{x}{2}\right)}$$

Divide num, and deno. by $\cos^2\left(\frac{x}{2}\right)$

$$= \int_0^\pi \frac{\sec^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}dx$$

$$= \frac{1}{a-b}\int_0^\pi \frac{\sec^2\frac{x}{2}}{\sqrt{\frac{(a+b)}{(a-b)}}^{\,2} + \tan^2\frac{x}{2}}dx$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\left(\tan\left(\frac{\frac{x}{2}}{\sqrt{\frac{a-b}{a+b}}}\right)\right)\right]_0^\pi$$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

$$= \frac{\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \qquad (1)$$

Diff. w.r.t. a by using Leibnitz's rule

$$\int_0^\pi \frac{-dx}{(a+b\cos x)^2} = \frac{\pi\left(-\frac{1}{2}\right)2a}{(a^2-b^2)^{3/2}}$$

$$\Rightarrow \int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$$

Which is the first deduction

Now we diff. (1) w.r.t. b by using Leibnitz's rule

$$\int_0^\pi \frac{-\cos x}{(a+b\cos x)^2}dx = \frac{\pi\left(-\frac{1}{2}\right)(-2b)}{(a^2-b^2)^{3/2}}$$

or $\int_0^\pi \frac{\cos x}{(a+b\cos x)^2}dx = \frac{-\pi b}{(a^2-b^2)^{3/2}}$

3. By applying differentiation under the Integral sign, prove that

$$\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi\left[\sqrt{1+y}-1\right] \text{ if } y>-1.$$

(September 2012)

Sol. We have $\phi(y) = \int_0^{\frac{\pi}{2}} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx$ **(1)**

Diff. w.r.t y be applying Leibnitz's rule

$$\frac{d}{dy}\phi(y) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial y}\frac{\log(1+y\sin^2 x)}{\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\frac{\sin^2 x}{1+y\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+y\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + y\tan^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{1 + \tan^2 x + y\tan^2 x} dx$$

Put $\tan x = t$
$\therefore \sec^2 x dx = dt$
$x = 0 \Rightarrow t = 0$
$x = \frac{\pi}{2} \Rightarrow t = \infty$

$$= \int_0^{\infty} \frac{1}{1+(1+y)t^2} dt$$

$$= \frac{1}{1+y}\int_0^{\infty} \frac{1}{\left(\frac{1}{1+y}\right)+t^2} dt = \frac{1}{1+y}\int_0^{\infty} \frac{dt}{\left(\sqrt{\frac{1}{1+y}}\right)^2+t^2}$$

$$= \frac{1}{1+y}\sqrt{1+y}\cdot\tan^{-1}\left[\frac{t}{1}\cdot\frac{\sqrt{1+y}}{1}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{1+y}}\left\{\tan^{-1}\infty - \tan^{-1}0\right\} = \frac{1}{\sqrt{1+y}}\cdot\frac{\pi}{2}$$

Integrating both sides w.r.t y

$\phi(y) = \pi\sqrt{1+y} + c$ **(2)**

For $y = 0$,

From (1) $\phi(0) = 0$

From (2) $\phi(0) = \pi + c \Rightarrow c = -\pi$

$\therefore$ From (2)

$\phi(y) = \pi\sqrt{1+y} - \pi$

$= \pi\left(\sqrt{1+y} - 1\right)$

4. Prove that $\int_0^{\pi/2} \log\left(\alpha\cos^2\theta + \beta\sin^2\theta\right)d\theta = \pi\log\left[\frac{1}{2}\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right]$ **where** $\alpha, \beta > 0$.

(April 2012)

Sol. Here α, β are arbitrary constants

Let $\phi(\alpha,\beta) = \int_0^{\frac{\pi}{2}} \log\left(\alpha\cos^2\theta + \beta\sin^2\theta\right)d\theta$ **(1)**

Diff. w.r.t. α by applying Leibnitz's rule

$$\frac{\partial}{\partial\alpha}\phi(\alpha,\beta) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial\alpha}\log\left(\alpha\cos^2\theta + \beta\sin^2\theta\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\alpha\cos^2\theta + \beta\sin^2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\alpha + \beta\tan^2\theta} d\theta$$

$$= \int_0^{\infty} \frac{dt}{\left(\alpha+\beta t^2\right)\left(1+t^2\right)} \quad (1)$$

Put $\tan\theta = t$
$\sec^2\theta d\theta = dt$
$\Rightarrow d\theta = \frac{dt}{1+t^2}$
$\theta = 0 \Rightarrow t = 0$
$\theta = \frac{\pi}{2} \Rightarrow t \to \infty$

Let us resolve $\frac{1}{\left(a+\beta t^2\right)\left(1+t^2\right)}$ into partial fractions,

for it use $t^2 = y$

$$\frac{1}{(\alpha+\beta y)(1+y)} = \frac{\beta}{\beta-\alpha}\cdot\frac{1}{\alpha+\beta y} + \frac{1}{\alpha-\beta}\cdot\frac{1}{1+y}$$

$$= \frac{\beta}{\beta-\alpha}\cdot\frac{1}{\alpha+\beta t^2} + \frac{1}{\alpha-\beta}\cdot\frac{1}{1+t^2}$$

$\therefore$ From (1)

$$\frac{\partial}{\partial\alpha}\phi(\alpha,\beta) = \frac{1}{\beta-\alpha}\int_0^{\infty}\left[\frac{\beta}{\alpha+\beta t^2} - \frac{1}{t^2+1}\right]dt$$

$$= \frac{1}{\beta-\alpha}\left[\int_0^{\infty}\frac{dt}{\frac{\alpha}{\beta}+t^2} - \int_0^{\infty}\frac{dt}{t^2+1}\right]$$

$$=\frac{1}{\beta-\alpha}\left[\frac{1}{\sqrt{\frac{\alpha}{\beta}}}\left(\tan^{-1}\frac{t}{\sqrt{\frac{\alpha}{\beta}}}\right)_0^{\infty}-\left(\tan^{-1}t\right)_0^{\infty}\right]$$

$$=\frac{1}{\beta-\alpha}\left[\sqrt{\frac{\beta}{\alpha}}\left(\tan^{-1}\infty-\tan^{-1}0\right)-\left(\tan^{-1}\infty+\tan^{-1}0\right)\right]$$

$$=\frac{1}{\beta-\alpha}\left[\sqrt{\frac{\beta}{\alpha}}\frac{\pi}{2}-0-\frac{\pi}{2}+0\right]$$

$$=\frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\alpha}(\beta-\alpha)}.\frac{\pi}{2}=\frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\alpha}\left(\sqrt{\beta}+\sqrt{\alpha}\right)\left(\sqrt{\beta}-\sqrt{\alpha}\right)}\frac{\pi}{2}$$

$$=\frac{\pi}{2}.\frac{1}{\sqrt{\alpha}\left(\sqrt{\beta}+\sqrt{\alpha}\right)}$$

Integrating w.r.t. α

$$\phi(\alpha,\beta)=\pi\int\frac{\frac{1}{2\sqrt{\alpha}}}{\sqrt{\alpha}+\sqrt{\beta}}d\alpha+c=\pi\log\left(\sqrt{\alpha}+\sqrt{\beta}\right)+c \quad \textbf{(2)}$$

Where c is constant of integration

To find c

Put $\alpha=1,\beta=1$ in (1) and (2)

From (1) $\phi(1,1)=\int_0^{\frac{\pi}{2}}\log\left(\cos^2\theta+\sin^2\theta\right)d\theta$

$$=\int_0^{\frac{\pi}{2}}\log 1 d\theta$$

$$=0$$

From (2) $\Rightarrow\phi(1,1)=\pi\log(1+1)+c$

$$=\pi\log 2+c$$

$$\therefore\ 0=\pi\log 2+c$$

$$c=-\pi\log 2$$

Now (2) $\Rightarrow\phi(\alpha,\beta)=\pi\log\left(\sqrt{\alpha}+\sqrt{\beta}\right)-\pi\log 2$

$=\pi\log\left(\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}\right)$ is required value of given integral.

5. Prove that $\int_0^{\infty}\frac{e^{-ax}\sin bx}{x}dx=\tan^{-1}\frac{b}{a}$ for a, b > 0 and deduce $\int_0^{\infty}\frac{\sin bx}{x}dx=\frac{\pi}{2}$.

(September 2011, April 2010)

Sol. The given integral contains two parameters 'a' and 'b'

$$F(a,b)=\int_0^{\infty}\frac{e^{-ax}\sin bx}{x}dx \quad \textbf{(1)}$$

Diff. w.r.t. b by using Leibnitz's rule

$$\frac{d}{db}\left[F(a,b)\right]=\int_0^{\infty}\frac{\partial}{\partial b}\left(e^{-ax}\sin bx.\frac{1}{x}\right)dx$$

$$=\int_0^{\infty}\frac{e^{-ax}(\cos bx)x}{x}dx$$

$$=\int_0^{\infty}e^{-ax}\cos bx dx$$

$$=\frac{e^{-ax}}{a^2+b^2}\left[-a\cos bx+b\sin bx\right]_0^{\infty}$$

$$=\frac{1}{a^2+b^2}\left[0-e^0(-a)\right]$$

$$=\frac{a}{a^2+b^2}$$

$$\Rightarrow F'(a,b)=\frac{a}{a^2+b^2}$$

Integrating w.r.t. 'b'

$$F(a,b)=a.\frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right)+c$$

$$= \tan^{-1}\left(\frac{b}{a}\right) + c \quad (2)$$

To find c

We take b = 0

From (2)

$F(a,0) = \tan^{-1} 0 + c = c$

From (1)

$$F(a,0) = \int_0^\infty e^{-ax}\frac{\sin 0x}{x}dx = 0$$

$\therefore$ c = 0

$$\therefore\ F(a,b) = \tan^{-1}\left(\frac{b}{a}\right)$$

i.e. $\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx = \tan^{-1}\left(\frac{b}{a}\right)$

Deduction: Put a = 0

$$\int_0^\infty \frac{\sin bx}{x}dx = \underset{a\to 0}{\text{Lt}} \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\infty = \frac{\pi}{2}.$$

6. **If $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous function of x and y for $a \le x \le b \,\&\, c \le y \le d$, then show**

$$\frac{d}{dy}\left[\int_a^b f(x,y)dx\right] = \left[\int_a^b f_y(x,y)dx\right] = \int_a^b \frac{\partial}{\partial y}f(x,y)dx$$ **(April 2011)**

Sol. Let $\phi(y) = \int_a^b f(x,y)dx \qquad \forall y \in [c,d] \quad (1)$

$$\Rightarrow \phi(y+k) - \phi(y) = \int_a^b \left(f(x,y+k) - f(x,y)\right)dx \quad (2)$$

By Lagrange's Mean Value theorem, we have

$$f(x,y+k) - f(x,y) = kf_y(x,y+\theta k), \qquad 0<\theta<1$$

$$= k\left[f_y(x,y+\theta k) - f_y(x,y) + f_y(x,y)\right] \quad (3)$$

Thus

$$\frac{\phi(y+k)-\phi(y)}{k} - \int_a^b f_y(x,y)dx$$

$$= \int_a^b \left[f_y(x,y+\theta k) - f_y(x,y)\right]dy \quad (4)$$

[Using (3)]

Since f_y is continuous in [a, b; c, d] $\in R^2$

$\therefore$ It is uniformly continuous

$\therefore$ Given $\varepsilon > 0$ $\exists$ a +ve number δ such that

$$\left|f_y(x,y+\theta k) - f_y(x,y)\right| < \frac{\varepsilon}{b-a} \quad for\ |k|<\delta \quad (5)$$

Hence from (4) and (5), we get

$$\left|\frac{\phi(y+k)-\phi(y)}{k} - \int_a^b f_y(x,y)\right| = \left|\int_a^b \left[f_y(x,y+\theta k) - f_y(x,y)\right]dy\right|$$

$$\le \int_a^b \left|f_y(x,y+\theta k) - f_y(x,y)\right|dy < \frac{\varepsilon}{b-a}(b-a) = \varepsilon \text{ when } |k|<\delta$$

$$\Rightarrow \lim_{k\to 0}\frac{\phi(y+k)-\phi(y)}{k} = \int_a^b f_y(x,y)$$

$\therefore$ $\phi(y)$ is differentiable

and $\frac{d}{dy}\phi(y) = \int_a^b f_y(x,y)$

i.e. $\frac{d}{dy}\left[\int_a^b f(x,y)dx\right] = \int_a^b f_y(x,y)dx$

7. **By applying differentiation under the integral sign prove**

$$\int_0^\infty e^{-xy}\frac{\sin x}{x}dx = \cot^{-1} y \text{ for } y > 0.$$ **(April 2011)**

Sol. The given integral contains one parameter 'y'

$$F(y) = \int_0^\infty e^{-xy}\frac{\sin x}{x}dx$$

Diff. w.r.t. y be using Leibnitz's Rule

$$\frac{d}{dy}\left[F(y)\right]=\int_0^\infty \frac{\partial}{\partial y}\left(\frac{e^{-xy}\sin x}{x}\right)dx$$

$$=\int_0^\infty \frac{-xe^{-xy}\sin x}{x}dx=-\int_0^\infty e^{-xy}\sin x dx$$

$$=-\left[\frac{e^{-xy}}{1+y^2}(-y\sin x-\cos x)\right]_0^\infty$$

$$=\left[\frac{e^{-xy}}{1+y^2}(y\sin x+\cos x)\right]_0^\infty$$

$$=\left[0-\frac{e^0}{1+y^2}(1)\right]=\frac{-1}{1+y^2}$$

$$\Rightarrow F'(y)=\frac{-1}{1+y^2}$$

Integrating w.r.t y

$$\Rightarrow F(y)=\cot^{-1}y$$

$$\Rightarrow \int_0^\infty e^{-xy}\frac{\sin x}{x}dx=\cot^{-1}y$$

8. **Given** $\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi}{2ab}$, **by differentiating under the integral sign deduce** $\int_0^{\pi/2}\frac{dx}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}=\frac{\pi\left(a^2+b^2\right)}{4a^3b^3}$. **(September 2010)**

Sol. We have $\int_0^{\pi/2}\frac{1}{a^2\cos^2 x+b^2\sin^2 x}dx=\frac{\pi}{2ab}$ **(1)**

Diff. both sides w.r.t. 'a' and using Leibnitiz's rule

$$\int_0^{\pi/2}\frac{-2a\cos^2 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}dx=-\frac{\pi}{2a^2b}$$

$$\Rightarrow \int_0^{\pi/2}\frac{\cos^2 x}{\left(a^2\cos^2 x+b^2\cos^2 x\right)^2}dx=\frac{\pi}{4a^3b} \quad \textbf{(2)}$$

Similarly diff. (1) w.r.t. 'b' using Leibnitz's rule

$$\int_0^{\pi/2}\frac{\sin^2 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}dx=\frac{\pi}{4ab^3} \quad \textbf{(3)}$$

Adding (2) and (3)

$$\int_0^{\pi/2}\frac{\cos^2 x+\sin^2 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}dx=\frac{\pi}{4a^3b}+\frac{\pi}{4ab^3}$$

$$\therefore \int_0^{\pi/2}\frac{1}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}dx=\frac{\pi}{4ab}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{\pi\left(a^2+b^2\right)}{4a^3b^3}$$

9. **Prove that if** $a>b$, $\int_0^{\frac{\pi}{2}}\log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta}=\pi\sin^{-1}\frac{b}{a}$. **(April 2009)**

Sol. In the given integrand, there are a and b two parameters. Because of the presence of $\frac{1}{\sin\theta}$ as a factor in integrand, we consider only 'b' as a parameter

Let $\phi(b)=\int_0^{\pi/2}\log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta}$ **(A)**

Diff. both sides w.r.t b by applying Leibnitz's rule

$$\frac{d}{db}(\phi(b))=\int_0^{\pi/2}\frac{\partial}{\partial b}\log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta}$$

$$=\int_0^{\pi/2}\frac{\partial}{\partial b}\left[\log(a+b\sin\theta)-\log(a-b\sin\theta)\right]\frac{d\theta}{\sin\theta}$$

$$=\int_0^{\pi/2}\left[\frac{\sin\theta}{a+b\sin\theta}+\frac{\sin\theta}{a-b\sin\theta}\right]\frac{d\theta}{\sin\theta}$$

$$=2a\int_0^{\pi/2}\frac{1}{a^2-b^2\sin^2\theta}d\theta$$

$$=2a\int_0^{\pi/2}\frac{\sec^2\theta}{a^2\sec^2\theta-b^2\tan^2\theta}d\theta$$

$$=2a\int_0^{\pi/2}\frac{\sec^2\theta}{a^2+\left(a^2-b^2\right)\tan^2\theta}d\theta$$

$$= 2a\int_0^\infty \frac{1}{a^2+\left(a^2-b^2\right)t^2}dt$$

Put $\tan\theta = 1 \Rightarrow \sec^2\theta d\theta = dt$

$\theta = 0 \Rightarrow t = 0$

$\theta = \frac{\pi}{2} \Rightarrow t \to \infty$

$$= \frac{2a}{a^2-b^2}\int_0^\infty \frac{1}{\left(\frac{a}{\sqrt{a^2-b^2}}\right)^2+t^2}dt$$

$$= \frac{2a}{a^2-b^2}\cdot\frac{\sqrt{a^2-b^2}}{a}.\tan^{-1}\left(\frac{t\sqrt{a^2-b^2}}{a}\right)\Bigg|_0^\infty$$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{\sqrt{a^2-b^2}}$$

Integrating, w.r.t. b

$$\phi(b) = \pi\sin^{-1}\left(\frac{b}{a}\right)+c \qquad \textbf{(B)}$$

Where c is constant of integration

To find c we put b = 0 in (A) and (B)

$$(A) \Rightarrow \phi(0) = \int_0^{\frac{\pi}{2}} log1\frac{d\theta}{\sin\theta} = 0$$

$$(B) \Rightarrow \phi(0) = \pi\sin^{-1}(0)+c = c$$

$\therefore$ c = 0

$$\therefore \int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta} = \pi\sin^{-1}\left(\frac{b}{a}\right)$$

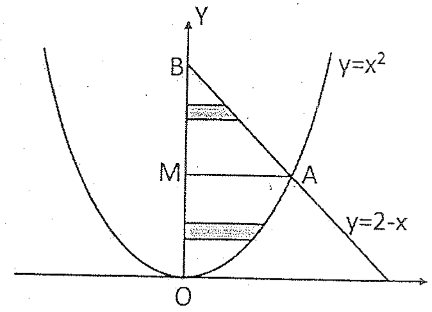
5

DOUBLE INTEGRALS AND ITS APPLICATIONS

1. **Change the order of integration and verify the result:** $\int_0^1\int_{x^2}^{2-x} xydydx$.

(September 2013, April 2012)

Sol. The region of integration is bounded by x = 0, x = 1 and $y = x^2$, y = 2 − x.
Now x = 0 is y-axis, x = 1 is a line parallel to y-axis.
Also $y = x^2$ is an upward parabola whose vertex is at (0, 0) and y = 2 − x is a straight line passing through point (0, 2), (1, 1) and (2, 0)
The parabola $y = x^2$ and the line y = 2 − x intersect at point (1, 1)
$\therefore$ region OAB is the region of integration.



In the given order we have to integrate first w.r.t. y and then w.r.t. x. After change of order of integration, we have to integrate first w.r.t. x and then w.r.t. y.
Divide region OAB into horizontal strips and for this we divide region OAB into two parts OAM and MAB.
In region OAM, each horizontal strip starts from line x = 0 and ends on curve $x = \sqrt{y}$.
Also these trips vary from y = 0 to y = 1. In region MAB, each horizontal strip starts from line x = 0 and ends on line x = 2 − y.
Also these strips vary form y = 1 to y = 2. Thus after change of order, we have

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{x}} xydxdy + \int_{y=1}^{2}\int_{x=0}^{2-y} xydxdy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{y}{2}(y)\,dy + \int_1^2 \frac{y}{2}(2-y)^2\,dy$$

$$= \int_0^1 \frac{y^2}{2}dy + \frac{1}{2}\int_1^2 y\left(4-4y+y^2\right)dy$$

$$= \int_0^1 \frac{y^2}{2}dy + \frac{1}{2}\int_1^2 \left(4y-4y^2+y^3\right)dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8-\frac{32}{3}+4\right)-\left(2-\frac{4}{3}+\frac{1}{4}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3}-\frac{11}{12}\right] = \frac{1}{6}+\frac{1}{2}\times\frac{5}{12} = \frac{9}{24}$$

2. Change the order of integration $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dydx$. Hence evaluate it when $f(x,y)=1$. **(April 2013)**

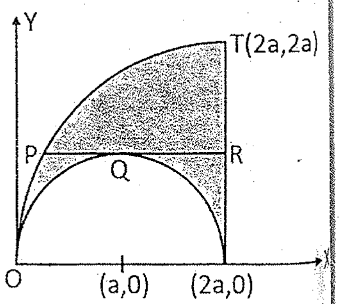
Sol. The region A of integration is given by

$A = \left\{(x,y): \sqrt{2ax-x^2} \le y \le \sqrt{2ax}, 0 \le x \le 2a\right\}$

$y=\sqrt{2ax-x^2}$ i.e. $x^2+y^2-2ax=0$ is circle with centre (a, 0) and radius a and $y=\sqrt{2ax}$ i.e. $y^2=2ax$ is right handed parabola with vertex at origin solving $y^2=2ax$ and x^2+y^2+2ax, we get $x^2=0$

i.e. x = 0 given y = 0

Cutting the region A by horizontal lines, we see that A is divided into three sub regions A_1, A_2, A_3 shown in figure by OPQ, QSR and PQRT respectively



Region A_1 (OPQ)

$= \left\{(x,y): \frac{y^2}{2a} \le x \le a-\sqrt{a^2-y^2}, 0 \le y \le a\right\}$

Region A_2 (QSR)

$= \left\{(x,y): a+\sqrt{a^2-y^2} \le x \le 2a; 0 \le y \le a\right\}$

Region A_3 (PQRT)

$= \left\{(x,y): \frac{y^2}{2a} \le x \le 2a;\ a \le y \le 2a\right\}$

$$\therefore \int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dydx = \iint_A f(x,y)\,dydx$$

$$= \iint_{A_1} f(x,y)\,dydx + \iint_{A_2} f(x,y)\,dydx + \iint_{A_3} f(x,y)\,dydx$$

$$= \int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dxdy + \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dxdy + \int_a^{2a}\int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dxdy$$

Here given f(x, y) = 1

$$\Rightarrow \int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} 1.dxdy = \int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} dxdy + \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} dxdy + \int_a^{2a}\int_{\frac{y^2}{2a}}^{2a} dxdy$$

$$= \int_0^a \left[a-\sqrt{a^2-y^2}-\frac{y^2}{2a}\right]dy + \int_0^a \left[2a-a-\sqrt{a^2-y^2}\right]dy + \int_a^{2a}\left[2a-\frac{y^2}{2a}\right]dy$$

$$= \left[ay-\frac{y}{2}\sqrt{a^2-y^2}-\frac{a^2}{2}\sin^{-1}\frac{y}{a}-\frac{y^3}{6a}\right]_0^a + \left[ay-\frac{y}{2}\sqrt{a^2-y^2}-\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a + \left[2ay-\frac{y^3}{6a}\right]_a^{2a}$$

$$= \left[a^2-\frac{a^2}{2}\left(\frac{\pi}{2}\right)-\frac{a^2}{6}\right] + \left[a^2-\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right] + \left[4a^2-\frac{4a^2}{3}-2a^2+\frac{a^2}{6}\right]$$

$$= \frac{5a^2}{6}-\frac{\pi a^2}{4}+a^2-\frac{\pi a^2}{4}+\frac{5a^2}{6}$$

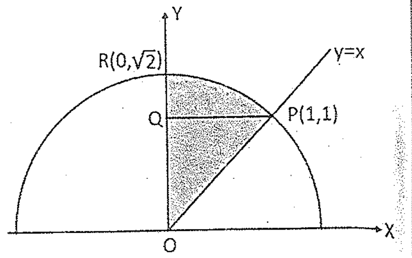
$$= \frac{8a^2}{3}-\frac{\pi a^2}{2}$$

$$= \left(\frac{16-3\pi}{6}\right)a^2$$

3. **Change the order of Integration in** $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ **and hence find the value.**

(September 2012)

Sol. Let $I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{xdydx}{\sqrt{x^2+y^2}}$



Here the region A of integration is given by

$A = \left\{(x,y): x \le y \le \sqrt{2-x^2}, 0 \le x \le 1\right\}$

$y = \sqrt{2-x^2}$ i.e. $x^2+y^2=2$ is circle with centre (0, 0) and radius $\sqrt{2}$

Solving y = x and $x^2+y^2=2$, we get $2y^2=2$ i.e. y = 1.

Hence the line y = x and semi circle $y=\sqrt{2-x^2}$ meet at (1, 1)

Region A can be divided into sub regions A_1 and A_2 shown respectively by OPQ and PRQ with help of horizontal lines

$\therefore\ A_1 = \{(x,y); 0 \le x \le y, 0 \le y \le 1\}$

$A_2 = \left\{(x,y); 0 \le x \le \sqrt{2-y^2}, 1 \le y \le \sqrt{2}\right\}$

$$\therefore\ I = \int_0^1\int_0^y \frac{xdxdy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{xdxdy}{\sqrt{x^2+y^2}}$$

$$= \int_0^1\int_0^y \frac{1}{2}(x^2+y^2)^{-1/2} 2xdxdy + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{1}{2}(x^2+y^2)^{-1/2} 2xdxdy$$

$$= \int_0^1 \left[(x^2+y^2)^{1/2}\right]_0^y dy + \int_1^{\sqrt{2}} \left[(x^2+y^2)^{1/2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 (\sqrt{2}y - y)dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)dy$$

$$= \left[(\sqrt{2}-1)\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \left[\frac{\sqrt{2}-1}{2} - 0\right] + \left[2 - 1 - \left(\sqrt{2} - \frac{1}{2}\right)\right]$$

$$= \frac{\sqrt{2}-1}{2} + 1 - \sqrt{2} + \frac{1}{2} = \frac{1}{2}\left(\sqrt{2} - 1 + 2 - 2\sqrt{2} + 1\right)$$

$$= \frac{2-\sqrt{2}}{2}$$

4. **Evaluate** $\iint (16 - x^2 - y^2)dxdy$ **over semi-circle** $x^2+y^2=4x$ **in the first quadrant.**

(September 2011)

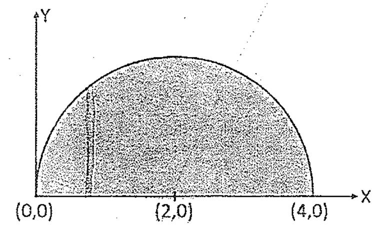
Sol. Here region of integration is

$x^2+y^2=4x$

$\Rightarrow (x^2-4x+y^2)=0$

$\Rightarrow (x^2-4x+4)+y^2=4$

$\Rightarrow (x-2)^2+y^2=2^2$



Which is a circle with centre (2, 0) Radius = 2

Here $0 \le y \le \sqrt{4x-x^2}$ and $0 \le x \le 4$

$$\iint_R (16-x^2-y^2)dxdy = \int_0^4 \int_0^{\sqrt{4x-x^2}} (16-x^2-y^2)dydx$$

$$= \int_0^4 \left[(16-x^2)y - \frac{y^3}{3}\right]_0^{\sqrt{4x-x^2}} dx$$

$$= \int_0^4 \left((16-x^2)\sqrt{4x-x^2} - \frac{(4x-x^2)^{3/2}}{3}\right)dx$$

$$= \int_0^4 \left(\sqrt{4x-x^2}\left(16-x^2-\frac{(4x-x^2)}{3}\right)\right)dx$$

$$= \frac{1}{3}\int_0^4 \sqrt{4x-x^2}\left(48-2x^2-4x\right)dx$$

Put $x = 4\sin^2\theta$ $\qquad x = 0 \Rightarrow \theta = 0$

$dx = 8\sin\theta\cos\theta d\theta$ $\qquad x = 4 \Rightarrow \theta = \frac{\pi}{2}$

$$\Rightarrow \iint_R (16-x^2-y^2)\,dxdy = \frac{1}{3}\int_0^{\pi/2} \sqrt{16\sin^2\theta - 16\sin^4\theta}\left[48-32\sin^4\theta-16\sin^2\theta\right] \times 8\sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} 4\sin\theta\cos\theta\left[48-32\sin^4\theta-16\sin^2\theta\right]8\sin\theta\cos\theta\, d\theta$$

$$= \frac{512}{3}\int_0^{\pi/2} \sin\theta\cos\theta\left(3-2\sin^4\theta-\sin^2\theta\right)\sin\theta\cos\theta\, d\theta$$

$$= \frac{512}{3}\int_0^{\pi/2}\left[3\sin^2\theta\cos^2\theta - 2\sin^6\theta\cos^2\theta - \sin^4\theta\cos^2\theta\right]d\theta$$

$$= \frac{512}{3}\left[3\left(\frac{1.1}{4.2}\cdot\frac{\pi}{2}\right) - 2\left(\frac{5.3.1.1}{8.6.4.2}\cdot\frac{\pi}{2}\right) - \left(\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}\right)\right]$$

$$= \frac{512\pi}{3}\left[\frac{24-5-4}{128}\right]$$

$$= \frac{512\pi}{3}\left[\frac{15}{128}\right] = (5\pi)(4) = 20\pi$$

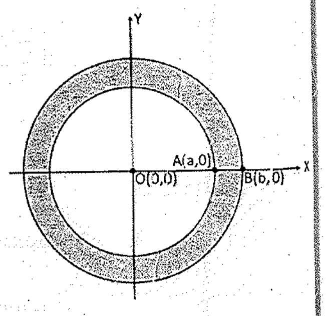
5. Evaluate $\iint_A (x^2+y^2)\,dxdy$ where A is region bounded by circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ where b > a. **(April 2011)**

Sol. Given curves are

$x^2+y^2=a^2$ and $x^2+y^2=b^2$ (b > a)

$x^2+y^2=a^2$ is circle with centre (0, 0) and radius = a

$x^2+y^2=b^2$ is circle with centre (0, 0) and radius = b



Using polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$

$a \le r \le b$, $|J| = r$

$$\therefore \iint_A (x^2+y^2)\,dxdy = 4\int_0^{\pi/2}\int_a^b (r^2)\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} d\theta \int_a^b r^3\,dr$$

$$= 4\left[\frac{\pi}{2}\right]\left[\frac{r^4}{4}\right]_a^b$$

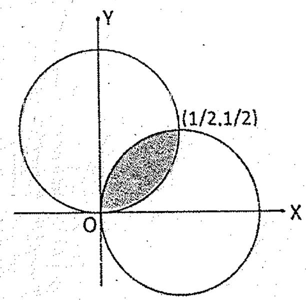
$$= \frac{\pi}{2}\left(b^4-a^4\right)$$

6. Show $\iint_A xy\,dx\,dy = \frac{1}{96}$ where A is the region common to circles $x^2+y^2=x$ and $x^2+y^2=y$. **(September 2010)**

Sol. The equation of two circles are

$x^2+y^2=x$ (1)

and $x^2+y^2=y$ (2)



From (1) and (2), we get

X = y

Putting x = y in (1), we get

$y^2+y^2=y$ or $2y^2-y=0$

$\therefore\ y(2y-1)=0$

$\Rightarrow y = 0, \frac{1}{2}$

$\therefore\ x = 0, \frac{1}{2}$

$\therefore$ circle (1) and (2) intersect in points (0, 0) and $\left(\frac{1}{2},\frac{1}{2}\right)$.

Now from (1), $y = \pm\sqrt{x-x^2}$

and from (2), $y = \frac{1\pm\sqrt{1-4x^2}}{2}$

But $0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2}$

$\therefore$ y varies from $\frac{1-\sqrt{1-4x^2}}{2}$ to $\sqrt{x-x^2}$

$$\therefore\ A = \left\{(x,y) : \frac{1-\sqrt{1-4x^2}}{2} \le y \le \sqrt{x-x^2}, 0 \le x \le \frac{1}{2}\right\}$$

Now $f(x,y)=xy$ is continuous over A

$$\therefore \iint_A xy\,dx\,dy=\int_0^{\frac{1}{2}} x\left(\int_{\frac{1-\sqrt{1-4x^2}}{2}}^{\sqrt{x-x^2}} y\,dy\right)dx$$

$$=\int_0^{\frac{1}{2}} x\left[\frac{y^2}{2}\right]_{\frac{1-\sqrt{1-4x^2}}{2}}^{\sqrt{x-x^2}} dx$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}} x\left[\left(\sqrt{x-x^2}\right)^2-\left(\frac{1-\sqrt{1-4x^2}}{2}\right)^2\right]dx$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}} x\left[-\frac{1+(1-4x^2)-2\sqrt{1-4x^2}}{4}+(x-x^2)\right]dx$$

$$=\frac{1}{8}\int_0^{\frac{1}{2}} x\left[-\left\{1+1-4x^2-2\sqrt{1-4x^2}\right\}+4x-4x^2\right]dx$$

$$=-\frac{1}{8}\int_0^{\frac{1}{2}} x\left[2-4x-2\sqrt{1-4x^2}\right]dx$$

$$=-\frac{1}{8}\int_0^{\frac{1}{2}}\left[2x-4x^2+\frac{1}{4}(1-4x^2)^{\frac{1}{2}}(-8x)\right]dx$$

$$=-\frac{1}{8}\int_0^{\frac{1}{2}}\left[\frac{2x^2}{2}-\frac{4x^3}{3}+\frac{1}{4}\frac{(1-4x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$

$$=-\frac{1}{8}\left[x^2-\frac{4}{3}x^3+\frac{1}{6}(1-4x^2)^{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$

$$=-\frac{1}{8}\left[\left(\frac{1}{4}-\frac{1}{6}+0\right)-\left(0-0+\frac{1}{6}\right)\right]$$

$$=-\frac{1}{8}\left[\frac{1}{4}-\frac{1}{3}\right]=-\frac{1}{8}\left[\frac{3-4}{12}\right]=\frac{1}{96}$$

7. Show $\iint_E \sqrt{x^2+y^2}\,dx\,dy=\frac{38\pi}{3}$ where E is region in xy-plane bounded by circles $x^2+y^2=4$ and $x^2+y^2=9$. **(April 2010)**

Sol. Given curves are

$x^2+y^2=4$ (1)

and $x^2+y^2=9$ (2)

Curve (1) is circle with centre (0, 0) and radius = 2

Curve (2) is circle with centre (0, 0) and radius = 3

Using polar co-ordinate

$x=r\cos\theta,\quad y=r\sin\theta$ we get

From (1) $r^2\cos^2\theta+r^2\sin^2\theta=4 \Rightarrow r=2$

From (2) $r^2\cos^2\theta+r^2\sin^2\theta=9 \Rightarrow r=3$

$\therefore 2\le r\le 3$

Also $|J|=\frac{\partial(x,y)}{\partial(r,\theta)}=r$

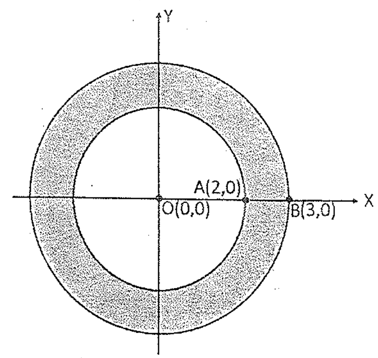
$$\iint_e \sqrt{x^2+y^2}\,dx\,dy=4\int_0^{\pi/2}\int_2^3 r(r\,dr\,d\theta)$$

$$=4\int_0^{\pi/2}d\theta\int_2^3 r^2\,dr$$

$$=4[\theta]_0^{\pi/2}\left[\frac{r^3}{3}\right]_2^3$$

$$=4\left[\frac{\pi}{2}\right]\left[\frac{27-8}{3}\right]$$

$$=\frac{38\pi}{3}$$



8. Evaluate $\iint_A x\,dx\,dy$ where A is the region bounded by parabolas $y^2=4ax$ and $x^2=4ay$. **(September 2009)**

Sol. Consider the equations

$y^2=4ax$ (1)

And $x^2=4ay$ (2)

From (1) and (2), we get

$$\left(\frac{y^2}{4a}\right)^2 = 4ay \Rightarrow y^4 - 64a^3y = 0$$

$$\Rightarrow y\left(y^3 - 64a^3\right) = 0 \Rightarrow y = 0, 4a$$

$$\therefore\ A = \left\{(x,y):\quad 0 \le y \le 4a,\quad \frac{y^2}{4a} \le x \le 2\sqrt{ay}\right\}$$

Now f(x, y) = x is continuous over A

$$\therefore\ \iint_A x\,dx\,dy = \int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} x\,dx\right]dy$$

$$= \int_0^{4a}\left[\frac{x^2}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left[2ay - \frac{y^4}{32a^2}\right]dy$$

$$= \left[ay^2 - \frac{y^5}{160a^2}\right]_0^{4a}$$

$$= 16a^3 - \frac{1024a^5}{160a^2}$$

$$= \left(16 - \frac{32}{5}\right)a^3$$

$$= \frac{48}{5}a^3$$

9. Evaluate $\iint_A x^2\,dx\,dy$ where A is the region enclosed by four parabolas $y^2 = ax$, $y^2 = bx$, $x^2 = cy$ and $x^2 = dy$ where a, b, c, d are +ve reals.

(April 2009)

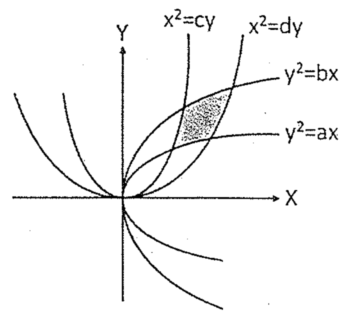
Sol. The region A of integration is bounded by the four parabolas $y^2 = ax$, $y^2 = bx$, $x^2 = cy$ and $x^2 = dy$.

Let us use the transformation

$\frac{y^2}{x} = u, \frac{x^2}{y} = v$ so that

the region A is mapped onto the region

$$B = \{(u,v):\ a \le u \le b,\ c \le v \le d\}$$



Now $\frac{y^2}{x} = u, \frac{x^2}{y} = v$

$$\Rightarrow \frac{x^4}{y^2}\cdot\frac{y^2}{x} = v^2u \quad \Rightarrow x^3 = uv^2 \Rightarrow x = u^{1/3}v^{2/3}$$

Also $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{vmatrix} = 1 - 4 = -3$

$$\therefore\ |J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$$

$$\therefore\ \iint_A x^2\,dx\,dy = \iint_B \left(u^{1/3}v^{2/3}\right)^2\left(\frac{1}{3}du\,dv\right)$$

$$= \frac{1}{3}\int_a^b u^{2/3}du \int_c^d v^{4/3}dv$$

$$= \frac{1}{3}\left[\frac{u^{\frac{5}{3}}}{\frac{5}{3}}\right]_a^b\left[\frac{v^{\frac{7}{3}}}{\frac{7}{3}}\right]_c^d$$

$$= \frac{1}{3}\cdot\frac{3}{5}\cdot\frac{3}{7}\left[b^{5/3} - a^{5/3}\right]\left[d^{7/3} - c^{7/3}\right]$$

$$= \frac{3}{35}\left(b^{5/3} - a^{5/3}\right)\left(d^{7/3} - c^{7/3}\right)$$

6

TRIPLE INTEGRALS AND ITS APPLICATIONS

1. Prove that $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz = \frac{\pi^2 abc}{4}$

where $V = \left\{(x,y,z) \middle| \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\right\}$. (September 2013, 2009)

Sol. Put $\frac{x}{a}=u, \frac{y}{b}=v, \frac{z}{c}=w, \quad \therefore x=au, y=bv, z=cw$

$\Rightarrow dx = adu,\ dy = bdv,\ dz = cdw$

$\therefore$ region of integration V transforms into

$V' = \{(u,v,w) : u^2+v^2+w^2 \le 1\}$

$$\text{Also } J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$$

$$= \iiint_{V'} \sqrt{1-u^2-v^2-w^2}\,.abc\,du\,dv\,dw$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} abc\sqrt{1-r^2}\,r^2 \sin\phi\, dr\, d\theta\, d\phi$$

$\left[\begin{array}{l}\text{Changing to spherical coordinates by substituting} \\ u = r\sin\phi\cos\theta, v = r\sin\phi\sin\theta, w = r\cos\theta \text{ and } |J| = r^2\sin\phi\end{array}\right]$

$$= abc\int_0^1 \sqrt{1-r^2}\,.r^2 dr \int_0^{2\pi} d\theta . \int_0^{\pi} \sin\phi\, d\phi$$

$$= abc.\frac{\pi}{16}.2\pi.2 = \frac{\pi^2 abc}{4}$$

$\left[\begin{array}{l} Let\ I = \int_0^1 \sqrt{1-r^2}.r^2 dr \\ Put\ r = \sin t, \quad \therefore dr = \cos t.dt \end{array}\right]$

$\left[\begin{array}{l} \therefore I = \int_0^{\pi/2} \sqrt{1-\sin^2 t}.\sin^2 t.\cos t dt = \int_0^{\pi/2} \sin^2 t.\cos^2 t dt \\ = \frac{1}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{16} \\ \int_0^{2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi - 0 = 2\pi \\ \int_0^{\pi} \sin\phi d\phi = [-\cos\phi]_0^{\pi} = -\cos\pi + \cos 0 = 1+1 = 2 \end{array}\right]$

2. **Prove that the volume of a right circular cone with base radius "a" and height "h" is $\frac{\pi a^2 h}{3}$ (using triple integration).** **(April 2013, September 2012)**

Sol. Consider the cross-section of the cone by a variable plane perpendicular to z-axis and passing through the point P(x, y, z) on the cone. Obviously it is a circle. Let OM = z, where M is the centre of the circle which in the case is the above mentioned cross-section of the cone. Obviously $0 \le z \le h$. Radius of the circle of section = MP = R (say). Now from similar triangle AMP and AOB, we have

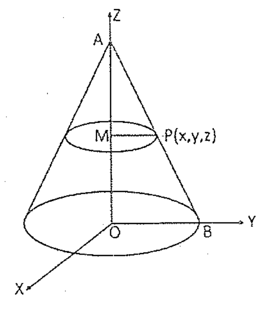
$$\frac{AM}{AO} = \frac{MP}{OB}$$

i.e. $\frac{h-z}{h} = \frac{R}{a}$

$[\because OA = h, OM = z,$

$\therefore AM = h - z$

and $OB = a, MP = R]$



$$\therefore\ R=\frac{a}{h}(h-z)$$

Since M lies on the z-axis where OM = z

$\therefore$ M = (0, 0, z) is the centre of the circle of section

$\therefore$ equation of the circle of section is

$$x^2+y^2=\left(\frac{a}{h}(h-z)\right)^2$$

Hence $V=\left\{(x,y,z);0\le z\le h, x^2+y^2\le\frac{a^2}{h^2}(h-z)^2\right\}$,

$$=\left\{(x,y,z);0\le z\le h,-\frac{a}{h}(h-z)\le y\le\frac{a}{h}(h-z)\right.$$

$$\left.-\sqrt{\frac{a^2}{h^2}(h-z)^2-y^2}\le x\le\sqrt{\frac{a^2}{h^2}(h-z)^2-y^2}\right\}$$

$\therefore$ Volume $=\iiint\limits_V 1.dx\,dy\,dz$

$$=\int_0^h\left(\int_{-\frac{a}{h}(h-z)}^{\frac{a}{h}(h-z)}\left(\int_{-\sqrt{\frac{a^2}{h^2}(h-z)^2-y^2}}^{\sqrt{\frac{a^2}{h^2}(h-z)^2-y^2}}1.dx\right)dy\right)dz$$

$$=2\int_0^h\left(\int_{-\frac{a}{h}(h-z)}^{\frac{a}{h}(h-z)}\sqrt{\frac{a^2}{h^2}(h-z)^2-y^2}\,dy\right)dz$$

$$=4\int_0^h\left(\int_0^{\frac{a}{h}(h-z)}\sqrt{\frac{a^2}{h^2}(h-z)^2-y^2}\,dy\right)dz$$

$$=4\int_0^h\frac{\pi}{4}\frac{a^2}{h^2}(h-z)^2\,dz$$

$$=\frac{\pi a^2}{h^2}\int_0^h(h-z)^2\,dz$$

$$=\frac{\pi a^2}{h^2}\left[\frac{(h-z)^3}{3(-1)}\right]_0^h$$

$$=\frac{1}{3}\pi a^2h$$

3. Evaluate $\iiint\limits_V\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}dx\,dy\,dz$ where

$$V=\left\{\begin{matrix}(x,y,z)\,x\ge0,\ y\ge0,\ z\ge0\\ x^2+y^2+z^2\le1\end{matrix}\right\}.$$

Sol. Since region of Integration is first octant

Put $x=r\cos\theta\sin\phi$

$y=r\sin\theta\sin\phi$

$y=r\cos\phi$ and $|J|=r^2\sin\phi$

Here $0\le r\le1,\quad 0\le\theta\le\frac{\pi}{2},\quad 0\le\phi\le\frac{\pi}{2}$

Consider $\iiint\limits_V\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}dxdydz$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1\frac{1-r^2}{1+r^2}r^2\sin\phi\,dr\,d\theta\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi\int_0^1\frac{(1-r^2)}{1+r^2}r^2\,dr$$

$$=[\theta]_0^{\frac{\pi}{2}}[-\cos\phi]_0^{\frac{\pi}{2}}\int_0^1\frac{r^2-r^4}{1+r^2}dr$$

$$=\left[\frac{\pi}{2}-0\right]\left[-\cos\frac{\pi}{2}+\cos0\right]\int_0^1\frac{r^2-r^4}{1+r^2}dr$$

$$=\frac{\pi}{2}\int_0^1\frac{r^2-r^4}{1+r^2}dr$$

$$=\frac{\pi}{2}\int_0^1\left[(2-r^2)-\frac{2}{1+r^2}\right]dr$$

$$=\frac{\pi}{2}\left[2r-\frac{r^3}{3}-2\tan^{-1}r\right]_0^1$$

$$=\frac{\pi}{2}\left[2-\frac{1}{3}-2\tan^{-1}(1)-0\right]$$

$$=\frac{\pi}{2}\left[\frac{5}{3}-\frac{\pi}{2}\right]$$

$$\Rightarrow \iiint_V \frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}dxdydz=\frac{\pi}{2}\left(\frac{5}{3}-\frac{\pi}{2}\right)$$

4. **Evaluate $\iiint_V \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ where V is the volume of sphere $x^2+y^2+z^2=1$ in the positive octant.** **(April 2012)**

Sol. Let $x=r\cos\theta\sin\phi$

$y=r\sin\theta\sin\phi$

$z=r\cos\phi \qquad \Rightarrow |J|=r^2\sin\phi$

Since region of Integration is first octant

$\Rightarrow 0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{2}$

Now $\iiint_V \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\phi drd\theta d\phi}{\sqrt{1-r^2}}$

$$=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\phi d\phi\int_0^1\frac{r^2dr}{\sqrt{1-r^2}}$$

$$=[\theta]_0^{\pi/2}[-\cos\phi]_0^{\pi/2}\int_0^1\frac{r^2dr}{\sqrt{1-r^2}}$$

$$=\left[\frac{\pi}{2}-0\right][0+1]\int_0^1\frac{r^2dr}{\sqrt{1-r^2}}=\frac{\pi}{2}\int_0^1\frac{r^2dr}{\sqrt{1-r^2}}$$

Put $r=\sin t\Rightarrow dr=\cos t dt$

$$=\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin^2 t\cos t dt}{\sqrt{1-\sin^2 t}}=\frac{\pi}{2}\int_0^{\pi/2}\sin^2 t dt$$

$$=\frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)=\frac{\pi^2}{8}$$

$$\Rightarrow \iiint_V\frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}=\frac{\pi^2}{8}$$

5. **Prove that $\iiint_{x^2+y^2+z^2\le 1}(z^5+z)dxdydz=0$** **(September 2011)**

Sol. Since $x^2+y^2+z^2\le 1$

$\therefore x^2\le 1, x^2+y^2\le 1, x^2+y^2+z^2\le 1$

Now $x^2\le 1 \qquad \Rightarrow -1\le x\le 1$

$x^2+y^2\le 1 \qquad \Rightarrow y^2\le 1-x^2$

$\Rightarrow -\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$

$x^2+y^2+z^2\le 1$

$\Rightarrow z^2\le 1-x^2-y^2$

$\Rightarrow -\sqrt{1-x^2-y^2}\le z\le\sqrt{1-x^2-y^2}$

$\therefore V=\left\{(x,y,z):-1\le x\le 1,-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\ -\sqrt{1-x^2-y^2}\le z\le\sqrt{1-x^2-y^2}\right\}$

$\therefore \iiint_{x^2+y^2+z^2\le 1}(z^5+z)dxdydz$

$$=\int_{-1}^{1}\left\{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}(z^5+z)dz\right)dy\right\}dx=0$$

$\left[\because\ f(z)=z^5+z \text{ is an odd function of z and } \therefore \int_{-a}^{a}f(z)dz=0\right]$

6. **Find volume of tetrahedron bounded by planes x = 0, y = 0, z = 0, $2x+3y+4z=1$.** **(September 2011)**

Sol. Required volume = $\iiint_V dxdydz$

where $V=\{(x,y,z);\ x\ge 0,\ y\ge 0,\ z\ge 0,\ 2x+3y+4z\le 1\}$

Put 2x = X, 3y = Y, 4z = Z

$dx=\frac{1}{2}dX,\quad dy=\frac{1}{3}dY,\quad dz=\frac{1}{4}dZ$

$\therefore V=\{(X,Y,Z);\ X\ge 0,\ Y\ge 0,\ Z\ge 0,\ X+Y+Z\le 1\}$

$=\{(X,Y,Z)\ 0\le Z\le 1,\ 0\le Y\le 1-Z,\ 0\le X\le 1-Y-Z\}$

Now $\iiint\limits_V dxdydz = \iiint\limits_V \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) dXdYdZ = \frac{1}{24}\int_0^1\int_0^{1-Z}\int_0^{1-Y-Z} dXdYdZ$

$= \frac{1}{24}\int_0^1\int_0^{1-Z} [X]_0^{1-Y-Z}\, dYdZ$

$= \frac{1}{24}\int_0^1 \left\{\int_0^{1-Z} (1-Y-Z)\,dY\right\} dZ$

$= \frac{1}{24}\int_0^1 \left[Y - \frac{Y^2}{2} - YZ\right]_0^{1-Z} dZ$

$= \frac{1}{24}\int_0^1 \left[(1-Z) - \frac{(1-Z)^2}{2} - Z(1-Z)\right] dZ$

$= \frac{1}{24}\int_0^1 \frac{(1-Z)^2}{2} dZ = \frac{1}{(24)(2)}\left[\frac{(1-Z)^3}{(-1)(3)}\right]_0^1$

$= \frac{1}{48}\left[0 + \frac{1}{3}\right] = \frac{1}{144}$

7. Evaluate $\iiint xyz\left(x^2+y^2+z^2\right)dxdydz$ over the positive octant of sphere $x^2+y^2+z^2=a^2$. **(April 2011)**

Sol. $V = \{(x,y,z):\ x \ge 0,\ \ y \ge 0,\ \ z \ge 0,\ \ x^2+y^2+z^2 \le a^2\}$

Put $x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\phi$

$V = \left\{(r,\theta,\phi):\quad 0 \le r \le a,\quad 0 \le \phi \le \frac{\pi}{2},\quad 0 \le \theta \le \frac{\pi}{2}\right\}$

$|J| = r^2\sin\phi$

$\therefore\ \iiint\limits_V xyz\left(x^2+y^2+z^2\right)dxdydz$

$= \iiint\limits_V (r\sin\phi\cos\theta)(r\sin\phi\sin\theta)(r\cos\phi)r^2\left(r^2\sin\phi\right)drd\theta d\phi$

$= \int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r^7\sin^3\phi\cos\phi\cos\theta\sin\theta\, d\theta dr d\phi$

$= \int_0^a r^7 dr \int_0^{\pi/2}\sin^3\phi\cos\phi d\phi \int_0^{\pi/2}\cos\theta.\sin\theta d\theta$

$= \left[\frac{r^8}{8}\right]_0^a \left[\frac{2.1}{4.2}\right]\left[\frac{1.1}{2.1}\right]$

$= \left(\frac{a^8}{8}\right)\left(\frac{1}{8}\right) = \frac{a^8}{64}$

8. Show $\int_1^e\int_1^{\log y}\int_1^{e^x}(\log z)\,dzdxdy = \frac{1}{4}\left(1+8e-3e^2\right)$. **(September 2010)**

Sol. Let $I = \int_1^e\int_0^{\log y}\int_1^{e^x}\log z\,dzdxdy$

$\therefore\ I = \int_1^e\left\{\int_0^{\log y}\left(\int_1^{e^x}\log z dz\right)dx\right\}dy \qquad (1)$

Now $\int_1^{e^x}\log z dz = \int_1^{e^x}\log z.1dz$

$= [\log z.z]_1^{e^x} - \int_1^{e^x}\frac{1}{z}.zdz$

$= \left(e^x\log e^x - \log 1\right) - \int_1^{e^x} 1dz = xe^x - [z]_1^{e^x}$

$= xe^x - \left(e^x - 1\right) = xe^x - e^x + 1$

$\therefore$ from (1), we get,

$I = \int_1^e\left\{\int_0^{\log y}\left(xe^x - e^x + 1\right)dx\right\}dy$

$= \int_1^e\left[xe^x - e^x - e^x + x\right]_0^{\log y} dy = \int_1^e\left[(x-2)e^x + x\right]_0^{\log y} dy$

$\left[\because \int xe^x dx = xe^x - \int 1.e^x dx = xe^x - e^x\right]$

$= \int_1^e\left\{(\log y - 2)e^{\log y} + \log y - (-2)\right\}dy$

$= \int_1^e\left\{y(\log y - 2) + \log y + 2\right\}dy$

$$= \int_1^e [(y+1)\log y - 2y + 2]\,dy$$

$$= \int_1^e (y+1).\log y dy - 2\int_1^e y dy + 2\int_1^e 1.dy$$

$$= \left[\log y.\left(\frac{y^2}{2}+y\right)\right]_1^e - \int_1^e \frac{1}{y}.\left(\frac{y^2}{2}+y\right)dy - 2\left[\frac{y^2}{2}\right]_1^e + 2[y]_1^e$$

$$= \left(\frac{e^2}{2}+e\right).\log e - 0 - \int_1^e \left(\frac{1}{2}y+1\right)dy - (e^2-1) + 2(e-1)$$

$$= \left(\frac{e^2}{2}+e\right).1 - \left[\frac{y^2}{4}+y\right]_1^e - (e^2-1) + 2(e-1)$$

$$= \frac{e^2}{2}+e - \left[\left(\frac{e^2}{4}+e\right) - \left(\frac{1}{4}+1\right)\right] - (e^2-1) + 2(e-1)$$

$$= \frac{e^2}{2}+e-\frac{e^2}{4}-e+\frac{5}{4}-e^2+1+2e-2 = -\frac{3e^2}{4}+2e+\frac{1}{4}$$

$$= \frac{1}{4}(1+8e-3e^2)$$

9. **Find volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.** **(September 2010)**

Sol. Required volume = $\iiint dxdydz$

Over the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Put $\frac{x}{a}=u, \frac{y}{b}=v, \frac{z}{c}=w \Rightarrow dx = a\,du, dy = b\,dv, dz = c\,dw$

$\therefore$ volume = $\iiint abc\,du\,dv\,dw$

Over the sphere $u^2+v^2+w^2=1$

Changing to spherical coordinates by the relations

$u = r\sin\phi\cos\theta,\ v = r\sin\phi\sin\theta,\ w = r\cos\phi$

$\Rightarrow |J| = r^2\sin\phi$

$\therefore$ volume = $abc\iiint r^2\sin\phi\,dr\,d\phi\,d\theta$

Over the region $\{(r,\phi,\theta): 0\le r\le 1, 0\le\phi\le\pi, 0\le\theta\le 2\pi\}$

$$= abc\int_0^1 r^2 dr\int_0^\pi \sin\phi\,d\phi\int_0^{2\pi} d\theta$$

$$= abc\left[\frac{r^3}{3}\right]_0^1 [-\cos\phi]_0^\pi [\theta]_0^{2\pi}$$

$$= abc\left(\frac{1}{3}-0\right)(-\cos\pi+\cos 0).(2\pi-0)$$

$$= abc\times\frac{1}{3}\times 2\times 2\pi = \frac{4\pi}{3}abc.$$

10. **Prove $\iiint_{x^2+y^2+z^2\le 1} z^2\,dxdydz = \frac{4\pi}{15}$.** **(April 2010)**

Sol. The region of integration is the interior of the unit sphere

$V = \{(x,y,z): \quad x^2+y^2+z^2\le 1\}$

$= \{(x,y,z): -1\le z\le 1, -\sqrt{1-z^2}\le y\le\sqrt{1-z^2},$

$-\sqrt{1-y^2-z^2}\le x\le\sqrt{1-y^2-z^2}\}$

$$\therefore \iiint_{x^2+y^2+z^2\le 1} z^2\,dxdydz = \int_{-1}^{1}\left[\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\left(\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} z^2\,dx\right)dy\right]dz$$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2z^2\sqrt{(1-z^2)-y^2}\,dy\right)dz$$

$$= \int_{-1}^{1} 2z^2\left[\frac{y\sqrt{1-z^2-y^2}}{2}+\frac{1-z^2}{2}\sin^{-1}\frac{y}{\sqrt{1-z^2}}\right]_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dz$$

$$= \int_{-1}^{1} 2z^2\left[0+\frac{1-z^2}{2}.\left(\frac{\pi}{2}\right)-0-\frac{1-z^2}{2}.\left(\frac{-\pi}{2}\right)\right]dz$$

$$= \int_{-1}^{1} 2z^2.2\frac{(1-z^2)}{2}.\frac{\pi}{2}dz$$

$$= \int_{-1}^{1} \pi z^2(1-z^2)\,dz$$

$$= 2\pi\int_0^1 z^2 - z^4\,dz$$

$$= 2\pi\left[\frac{z^3}{3}-\frac{z^5}{5}\right]_0^1$$

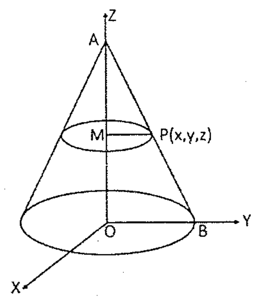
$$= 2\pi\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 2\pi\left(\frac{2}{15}\right) = \frac{4\pi}{15}$$

11. Find the volume of truncated cone with end radii a and b and height h.

(September 2009)

Sol. Consider the cross-section of the cone by a variable plane perpendicular to z-axis and passing through the point P(x, y, z) on the cone. Obviously it is a circle. Let OM = z, where M is the centre of the circle which in the case is the above mentioned cross-section of the cone. Obviously $0 \le z \le h$. Radius of the circle of section = MP = R (say). Now from similar triangle AMP and AOB, we have



$$\frac{AM}{AO} = \frac{MP}{OB}$$

i.e. $\frac{h-z}{h} = \frac{R}{a}$

$[\because OA = h, OM = z,$

$\therefore AM - h - z$

and $OB = a, MP = R]$

$$\therefore\ R = \frac{a}{h}(h-z)$$

Since M lies on the z-axis where OM = z

$\therefore$ M = (0, 0, z) is the centre of the circle of section

$\therefore$ equation of the circle of section is

$$x^2 + y^2 = \left(\frac{a}{h}(h-z)\right)^2$$

Hence $V = \left\{(x,y,z); 0 \le z \le h, x^2 + y^2 \le \frac{a^2}{h^2}(h-z)^2\right\}$,

$$= \left\{(x,y,z); 0 \le z \le h, -\frac{a}{h}(h-z) \le y \le \frac{a}{h}(h-z)\right.$$

$$\left. -\sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2} \le x \le \sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}\right\}$$

$\therefore$ Volume = $\iiint_V 1.dx\ dy\ dz$

$$= \int_0^h \left(\int_{-\frac{a}{h}(h-z)}^{\frac{a}{h}(h-z)} \left(\int_{-\sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}}^{\sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}} 1.dx \right) dy \right) dz$$

$$= 2\int_0^h \left(\int_{-\frac{a}{h}(h-z)}^{\frac{a}{h}(h-z)} \sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}\, dy \right) dz$$

$$= 4\int_0^h \left(\int_0^{\frac{a}{h}(h-z)} \sqrt{\frac{a^2}{h^2}(h-z)^2 - y^2}\, dy \right) dz$$

$$= 4\int_0^h \frac{\pi}{4}\frac{a^2}{h^2}(h-z)^2\, dz$$

$$= \frac{\pi a^2}{h^2}\int_0^h (h-z)^2\, dz$$

$$= \frac{\pi a^2}{h^2}\left[\frac{(h-z)^3}{3(-1)}\right]_0^h$$

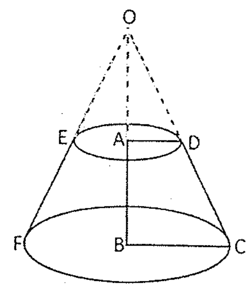
$$= \frac{1}{3}\pi a^2 h$$

$\Rightarrow$ volume of cone with radius r and height $h = \frac{1}{3}\pi r^2 h$

Given truncated cone with base radii 'a' and 'b' and height 'h'

$\therefore$ AD = a, BC = b, AB = h

Δ's OAD and OBC are similar



$\Rightarrow \frac{OA}{AD} = \frac{OB}{BC}$

Let OA = α

$\Rightarrow \frac{\alpha}{a} = \frac{\alpha + h}{b}$

$\Rightarrow \alpha b = \alpha a + ah$

$\Rightarrow \alpha = \frac{ah}{b-a} \Rightarrow OA = \frac{ah}{b-a}$

Volume of truncated cone

= Volume of cone OFC – Volume of cone OED

$$= \frac{1}{3}\pi b^2 (OB) - \frac{1}{3}\pi a^2 (OA)$$

$$= \frac{1}{3}\pi\left[b^2\left(\frac{ah}{b-a}+h\right) - a^2\left(\frac{ah}{b-a}\right)\right]$$

$$= \frac{1}{3}\pi\left[\frac{b^3h - a^3h}{b-a}\right]$$

$$= \frac{1}{3}\pi\frac{(b-a)(b^2+ab+a^2)}{b-a}h$$

$$= \frac{1}{3}\pi(b^2+ab+a^2)h$$

12. Show that $\iiint_V (x^2+y^2+z^2)^m \, dxdydz = \frac{4\pi}{2m+3}$ where V is the volume bounded by the sphere $x^2+y^2+z^2<1$. **(April 2009)**

Sol. Since the region of integration is a ball bounded by the sphere $x^2+y^2+z^2=1$, so we change to spherical coordinates by substituting $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$ and $|J| = r^2\sin\phi$

$\therefore$ V $= \{(r,\phi,\theta): 0\le r\le 1, 0\le\phi\le\pi, 0\le\theta\le 2\pi\}$

$$\therefore \iiint (x^2+y^2+z^2)^m \, dxdydz = \int_0^1\int_0^{2\pi}\int_0^{\pi}(r^2)^m . r^2 \sin\phi \, d\phi \, d\theta \, dr$$

$$[\because x^2+y^2+z^2 = r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi = r^2]$$

$$= \int_0^1 r^{2m+2}dr \int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi \, d\phi = \left[\frac{r^{2m+3}}{2m+3}\right]_0^1 [\theta]_0^{2\pi}[-\cos\phi]_0^{\pi}$$

$$= \left(\frac{1}{2m+3} - 0\right)(2\pi - 0)(-\cos\pi + \cos 0)$$

$$= \frac{1}{2m+3}\times 2\pi\times 2 = \frac{4\pi}{2m+3}$$

13. Find the volume of the tetraphedron bouded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. **(April 2009)**

Sol. Required volume $= \iiint_V dx \, dy \, dz$

Where V $= \left\{(x,y,z): x\ge 0, y\ge 0, z\ge 0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1\right\}$

Put $\frac{x}{a} = X, \frac{y}{b} = Y, \frac{z}{c} = Z$

$\therefore dx = a\,dX,\ dy = b\,dY,\ dz = c\,dZ$

$\therefore$ V $= \{(X,Y,Z): X\ge 0, Y\ge 0, Z\ge 0, X+Y+Z\le 1\}$

$= \{(X,Y,Z): 0\le Z\le 1, 0\le Y\le 1-Z, 0\le X\le 1-Y-Z\}$

Required volume $= \iiint_V dx \, dy \, dz$

$$= \iiint_V abc \, dX \, dY \, dZ = abc\int_0^1\int_0^{1-Z}\int_0^{1-Y-Z} dX \, dY \, dZ$$

$$= abc\int_0^1\int_0^{1-Z}[X]_0^{1-Y-Z} dY \, dZ$$

$$= abc\int_0^1\left\{\int_0^{1-Z}(1-Y-Z)\right\}dZ$$

$$= abc\int_0^1\left[Y - \frac{Y^2}{2} - YZ\right]_0^{1-Z} dZ$$

$$= abc\int_0^1\left[(1-Z) - \frac{(1-Z)^2}{2} - Z(1-Z)\right]dZ$$

$$= \frac{abc}{2}\int_0^1(1-Z)^2 \, dZ = \frac{abc}{2}\left[\frac{(1-Z)^3}{(-1)(3)}\right]_0^1$$

$$= -\frac{abc}{6}\left[(1-1)^3 - (1-0)^3\right] = \frac{abc}{6}$$

7

SEQUENCE AND SERIES OF FUNCTIONS

1. State Weierstrass's M-test for Series of functions. Hence show that series $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots$ converges uniformly on $\mathbb{R}$. **(September 2013)**

Sol. **Weierstrass's M-test:** A series $\sum_{n=1}^{\infty} U_n(x)$ of function will converge uniformly on X

(i) $|U_n(x)| \le M_n$ for all n and $\forall x \in X$ and

(ii) ΣM_n is convergent

Now $\Sigma U_n(x) = \cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots$

$\therefore U_n(x) = \frac{\cos nx}{n^2}$

$|U_n(x)| = \left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2} \quad \forall x \in R$

But $\Sigma \frac{1}{n^2}$ is cgt [Using p-test]

$\therefore$ By W. M. Test the given series is uniformly convergent in R

2. Show that if $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$: then it has a differential coefficient equal to

$$-2x\sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \forall x.$$

(September 2013)

Sol. Here $u_n(x) = \frac{1}{n^3 + n^4 x^2}$

$$u_n'(x) = -\frac{2}{n^2(1+nx^2)^2}$$

Now $u_n'(x)$ is maximum value $\frac{d}{dx}(u_n'(x)) = 0$

$\therefore (1+nx^2)^2 - 4nx^2(1+nx^2) = 0$

or $1 - 3nx^2 = 0$

or $x = \frac{1}{\sqrt{3n}}$

$\therefore Max.|u_n'(x)| = \frac{2}{\sqrt{3}n^{5/2}\left(1+\frac{1}{3}\right)^2}$

$= \frac{3\sqrt{3}}{8n^{5/2}}$

Then $|u_n'(x)| < \frac{1}{n^{5/2}}$ for all value of x.

But $\sum \frac{1}{n^{5/2}}$ is convergent.

Hence by W.M. Test, the series $\sum u_n'$ is U.C. for all real values of x. The term by term differentiable is therefore justified

$f'(x) = \sum_{n=1}^{\infty} u_n'(x)$

$= -2x\sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$.

3. Examine for term by term integration of the series whose sum of "n" terms is $n^2x(1-x)^n$. **(September, 2013, 2012, 2010, 2009, April 2010)**

Sol. Here $f_n(x) = n^2x(1-x)^n$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^2x(1-x)^n = \lim_{n\to\infty} \frac{n^2x}{(1-x)^{-n}} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{2nx}{(1-x)^{-n}\log(1-x)(-1)} \quad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{n\to\infty} \frac{2x}{(1-x)^{-n}\left[\log(1-x)\right]^2} = 0 \text{ where } 0 < x \le 1$$

$$\Rightarrow \int_0^1 f(x)\,dx = 0$$

$$\text{But } \int_0^1 f_n(x)\,dx = \int_0^1 n^2 x(1-x)^n\,dx = n^2\left[-\frac{x(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1$$

$$= \frac{n^2}{(n+1)(n+2)}$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n^2}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1$$

Since $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \ne \int_0^1\left(\lim_{n\to\infty} f_n(x)\right)dx$

So, term by term integration is not possible in $0 \le x \le 1$

4. **State and prove Abel's Test for uniform convergence of series.**

(April 2013, September 2013)

Sol. **Statement:–**

Let the series $\sum u_n(x)$ cgs uniformly in [a, b] and let sequence $\{v_n(x)\}$ be monotonic for each x in [a, b] and be uniformly bounded in [a, b]. Then the series $\sum u_n(x)\, v_n(x)$ is uniformly convergent in [a, b]

Proof:

Let $R_{n,p}(x)$ be partial remainder for the series $\sum u_n(x)\, v_n(x)$ and let S_n be the sum of n terms and $r_{n,p}(x)$ be the partial remainder for the series $\sum u_n(x)$.

$$R_{n,p} = u_{n+1}(x)\, v_{n+1}(x) + u_{n+2}(x) v_{n+2}(x) + \ldots\ldots + u_{n+p}(x) v_{n+p}(x)$$

$$= r_{n,1}(x) v_{n+1}(x) + \{r_{n,2}(x) - r_{n,1}(x)\} v_{n+2}(x) + \{r_{n,3}(x) - r_{n,2}(x)\} v_{n+3}(x) + \ldots\ldots + \{r_{n,p}(x) - r_{n,p-1}(x)\} v_{n+p}(x)$$

$$= r_{n,1}(x)\{v_{n+1}(x) - v_{n+2}(x)\} + r_{n,2}(x)\{v_{n+2}(x) - v_{n+3}(x)\} + \ldots\ldots + r_{n,p-1}(x)\{v_{n+p-1}(x) - v_{n+p}(x)\} + r_{n,p} v_{n+p}(x) \qquad (1)$$

Since $\{v_n(x)\}$ is monotonic

$\{v_{n+1}(x) - v_{n+2}(x)\}, \{v_{n+2}(x) - v_{n+3}(x)\}, \ldots\ldots, \{v_{n+p-1}(x) - v_{n+p}(x)\}$

have all of them the same sign for any fixed value of x in [a, b]

Also since $\{v_n(x)\}$ is uniformly bounded on [a, b].

$\therefore |r_{n,1}(x)|, |r_{n,2}(x)|, \ldots\ldots |r_{n,p}(x)|$ are each $< \frac{\varepsilon}{3K}$ when $n \ge m$,

the same m serving for all values of x in [a, b].

$$\therefore (1) \Rightarrow |R_{n,p}(x)| < \frac{\varepsilon}{3K}|v_{n+1}(x) - v_{n+p}x| + \frac{\varepsilon}{3K}|v_{n+p}(x)|$$

$$< \frac{\varepsilon}{3K}.2K + \frac{\varepsilon}{3K}.K = \varepsilon$$

$$\left[\because |v_{n+1}(x) - v_{n+p}(x)| \le |v_{n+1}(x)| + |v_{n+p}(x)| < K + K = 2K\right]$$

$\Rightarrow |R_{n,p}(x)| < \varepsilon$ when $n \ge m$ where m is fixed integer depending on ε but independent of x.

$\Rightarrow \sum u_n(x) v_n(x)$ cgs uniformly in [a, b].

5. **Prove that sequence $\{f_n(x)\}$ is not uniform convergent over real number where $f_n(x) = \frac{nx}{1+n^2x^2}$.**

(April 2013, 2010, September 2009)

Sol. Here

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{nx}{1+n^2x^2} = \lim_{n\to\infty}\frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} = \frac{0}{0+x^2} = 0$$

$$\Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}$$

Now $|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \left|\frac{nx}{1+n^2x^2}\right|$

Let $y = \frac{nx}{1+n^2x^2}$.

$$\frac{dy}{dx} = \frac{(1+n^2x^2)(n) - (nx)(2n^2x)}{(1+n^2x^2)^2} = \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

For max or min $\frac{dy}{dx} = 0$

$\Rightarrow n - n^3x^2 = 0 \quad \Rightarrow x^2 = \frac{1}{n^2} \Rightarrow x = \frac{1}{n}$

Now $\frac{dy}{dx} = \frac{n - n^3x^2}{(1+n^2x^2)^2}$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1+n^2x^2)^2(-2n^3x) - (n-n^3x^2)2(1+n^2x^2)(2n^2x)}{(1+n^2x^2)^4} = \frac{-2n^2x(3n^3x^2-n)}{(1+n^2x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n}} = \frac{-2n^2\left(\frac{1}{n}\right)\left(3n^3\frac{1}{n^2}-n\right)}{\left(1+n^2\frac{1}{n^2}\right)^3} = \frac{-2n(3n-n)}{(1+1)^3} = \frac{-4n^2}{8} = \frac{-n^2}{2} < 0$$

$\Rightarrow$ y is maximum, when $x = \frac{1}{n}$

$$y_{\max} = \frac{n.\frac{1}{n}}{1+n^2.\frac{1}{n^2}} = \frac{1}{1+1} = \frac{1}{2}$$

Thus $M_n = \underset{x\in[a,b]}{Sup}|f_r(x) - f(x)| = \underset{x\in[a,b]}{Sup}\left|\frac{nx}{1+n^2x^2} - 0\right| = \underset{x\in[a,b]}{Sup}\left|\frac{nx}{1+n^2x^2}\right| = \frac{1}{2}$

Thus, M_n does not tend to zero as n $\to\infty$

$\therefore$ By M_n-test sequence $< f_n >$ is not uniformly convergent in any interval containing zero.

6. Prove that the sum function of a uniformly convergent series of continuous functions is itself continuous. Is converse true? (April 2013)

Sol. Let the series $\sum u_n(x)$ be uniformly convergent in [a, b].

We are to prove that $f(x) = \sum_1^\infty u_n(x)$ is continuous in [a, b]

Let $f_n(x) = \sum_1^n u_n(x)$

Since the series is uniformly cgs in [a, b]

$\therefore$ for ε> 0, we can find a –ve integer 'm' independent of x s.t.

$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \qquad \forall n \geq m$ and $\forall x \in [a,b]$

In particular,

$|f_m(x) - f(x)| < \frac{\varepsilon}{3} \qquad \forall x$ in $[a,b]$ **(1)**

and $\qquad |f_m(c) - f(c)| < \frac{\varepsilon}{3}$ when $a \leq c \leq b$ **(2)**

Now $f_m(x)$ is the sum of a number m of continuous function and so is itself continuous.

$\therefore \exists\delta > 0$ s.t. $|f_m(x) - f_m(c)| < \varepsilon$ for all x in $|x-c| < \delta$ **(3)**

From (1), (2) and (3)

$|f(x) - f(c)| = |f(x) - f_m(x) + f_m(x) - f_m(c) + f_m(c) - f(c)|$

$\leq |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

Hence f(x) is cont. at any point c of [a, b]

$\therefore$ f(x) is cont. in [a, b].

It may be noted that uniform convergence is only a sufficient but not a necessary condition i.e. the converse of the above theorem is not true.

Consider the series $\sum\left[\frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}\right]$ in [a, b]

Here, $u_n(x) = \frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}$

$\therefore u_1 = \frac{x}{1+x^2} - 0$

$u_2 = \frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}$, ...

....

....

$u_n = \frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}$

Adding, we get $\quad f_n(x) = \frac{n^2x}{1+n^3x^2}$

$\Rightarrow f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{n^2x}{1+n^3x^2} = 0 \qquad \forall x \in [0,1]$

Thus $f(x) = 0 \qquad \forall x \in [0,1]$

Hence f(x) is continuous in [0, 1] $\therefore$ cont. at x = 0

Now $M_n = Sup\{|f_n(x) - f(x)| : x \in [0,1]\}$

$= Sup\left\{\left|\frac{n^2 x}{1+n^3 x^2} - 0\right| : x \in [0,1]\right\}$

$= Sup\left\{\left[\frac{n^2 x}{1-n^3 x^2}\right] : x \in [0,1]\right\} \geq \frac{\sqrt{n}}{2}$

$\left[\text{Taking } x = \frac{1}{n^{3/2}} \in [0,1]\right]$

$\Rightarrow M_n$ doesn't converge to 0 as $n \to \infty$

$\therefore$ The series $\sum_{1}^{\infty} u_n(x)$ does not converge uniformly in [0, 1] as $n \to \infty$

$\Rightarrow x = 0$ is a point of non-uniform convergence.

Hence the converse is not true.

7. **Test the series $\sum_{n=1}^{\infty}\left(\frac{n^2 x}{1+n^4 x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}\right)$ for uniform convergence in [0, 1].**

Can the series be integrated term by term? Justify. **(April 2013)**

Sol. Let $u_n = \frac{n^2 x}{1+n^4 x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}$

$\therefore f_n(x) = u_1 - u_2 + \ldots\ldots + u_n$

$\Rightarrow f_n(x) = \left(\frac{x}{1+x^2} - 0\right) + \left(\frac{2^2 x}{1+2^4 x^2} - \frac{x}{1+x^2}\right) + \ldots\ldots + \left(\frac{n^2 x}{1+n^4 x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}\right)$

$= \frac{n^2 x}{1+n^4 x^2}$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall x$

$\therefore$ the sum function is continuous $\forall x \in [0,1]$

Now $M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$

$= \sup\left\{\left[\frac{n^2 x}{1+n^4 x^2}\right] : x \in [0,1]\right\} = \frac{1}{2}$

Taking $x = \frac{1}{n^2} \in [0,1]$

$\Rightarrow M_n$ doesn't converge to 0 as $n \to \infty$

$\therefore$ The series $\sum_{n=1}^{\infty} u_n(x)$ doesn't converge uniformly in [0, 1] as $n \to \infty$

x = 0 is point of non-uniform convergence

Now $u_n(x) = \frac{n^2 x}{1+n^4 x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}$

Here $f_n(x) = \frac{n^2 x}{1+n^4 x^2}$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0,1]$

Hence $\int_0^1 \left(\sum_{n=1}^{\infty} u_n(x)\right) dx = \int_0^1 f(x)\,dx = \int_0^1 0.dx = 0$

and $\sum_{n=1}^{\infty} \int_0^1 u_n(x)\,dx = \lim_{n\to\infty} \int_0^1 \left[\sum_{n=1}^{n} u_n(x)\right] dx$

$= \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$

$= \lim_{n\to\infty} \int_0^1 \frac{n^2 x}{1+n^4 x^2} dx = \lim_{n\to\infty} \frac{1}{2n^2}\left[\log(1+n^4 x^2)\right]_0^1$

$= \lim_{n\to\infty} \frac{1}{2n^2} \log(1+n^4) \qquad \left[\frac{\infty}{\infty} form\right]$

$= \lim_{n\to\infty} \frac{\frac{4n^3}{1+n^4}}{4n} = \lim_{n\to\infty} \frac{n^2}{1+n^4} \qquad \left[\frac{\infty}{\infty} form\right]$

$= \lim_{n\to\infty} \frac{2n}{4n^3} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$

Thus $\sum_{n=1}^{\infty} \int_0^1 u_n(x)\,dx = \int_0^1 \sum_{n=1}^{\infty} u_n(x)\,dx$

Although x = 0 is a point of non-uniform convergence of the series

8. **Show that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots$ is not uniformly convergent on [0, 1].** **(September 2012)**

Sol. Let $S_n(x)$ be sum of n-term of series i.e. $S_n(x) = x^4 + \frac{x^4}{(1+x^4)^1} + \ldots + \frac{x^4}{(1+x^4)^{n-1}} = \frac{x^4\left[1-\frac{1}{(1+x^4)^n}\right]}{1-\frac{1}{1+x^4}}$

$$= \frac{x^4\left[1-\frac{1}{(1+x^4)^n}\right]}{\frac{1+x^4-1}{(1+x^4)}} = (1+x^4)\left[1-\frac{1}{(1+x^4)^n}\right]$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^4 & \text{when } 0 < x \le 1 \\ 0 & \text{when } x = 0 \end{cases}$$

Since S(x) exists for all value of x in [0, 1]. Then, the series is convergent in [0, 1]

For $0 < x \le 1$ and For given ε> 0

$$|S_n(x) - S(x)| = \left|(1+x^4)\left[1-\frac{1}{(1+x^4)^n}\right] - (1+x^4)\right|$$

$$= \left|\frac{-(1+x^4)}{(1+x^4)^n}\right| = \frac{1+x^4}{(1+x^4)^n} = \frac{1}{(1+x^4)^{n-1}} < \varepsilon \text{ if}$$

$$(1+x^4)^{n-1} > \frac{1}{\varepsilon}$$

if $\log(1+x^4)^{n-1} > \log\frac{1}{\varepsilon}$

if $n-1 > \frac{\log\frac{1}{\varepsilon}}{\log(1+x^4)}$

i.e., if $\quad n > 1 + \frac{\log\frac{1}{\varepsilon}}{\log(1+x^4)} \Rightarrow n \to \infty$ if $x \to 0$

So, the given series is not uniform convergent in [0, 1] and x = 0 is point of non-uniform convergence of series.

9. Test for uniform convergence of the sequence $\left\{\frac{n^2x}{1+n^3x^2}\right\}$ on the interval [0, 1].

(September 2012)

Sol. Let $f_n(x) = \frac{n^2x}{1+n^3x^2}$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^3}+x^2} = 0 \qquad \forall x \in [0,1]$$

$\Rightarrow$ f(x) = 0 ∀ x

Now $|f_n(x) - f(x)| = \left|\frac{n^2x}{1+n^3x^2} - 0\right| = \left|\frac{n^2x}{1+n^3x^2}\right|$

Let $y = \frac{n^2x}{1+n^3x^2}$

$$\frac{dy}{dx} = \frac{(1+n^3x^2)(n^2)-(n^2x)(2n^3x)}{(1+n^3x^2)^2} = \frac{n^2-n^5x^2}{(1+n^3x^2)^2}$$

For max. or min. $\frac{dy}{dx} = 0$

$\Rightarrow n^2 - n^5x^2 = 0 \qquad \Rightarrow x = \frac{1}{n^{3/2}}$

$$\frac{d^2y}{dx^2} = \frac{(1+n^3x^2)^2(-2n^5x)-(n^2-n^5x^2)2(1+n^3x^2)2n^3x}{(1+n^3x^2)^4} = \frac{-2n^3x(3n^2-n^5x^2)}{(1+n^3x^2)^3}$$

(On simplification)

$$\text{Now } \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n^{3/2}}} = \frac{-2n^3\left(\frac{1}{n^{\frac{3}{2}}}\right)\left(3n^2-n^5.\frac{1}{n^3}\right)}{\left(1+n^3.\frac{1}{n^3}\right)^2} = \frac{-n^{\frac{3}{2}}}{2}.2n^2 = -n^{\frac{7}{2}} < 0$$

Which show y is maximum at $x = \frac{1}{n^{3/2}}$

$$y = \frac{n^2x}{1+n^3x^2}$$

$$(y)_{max} = \frac{n^2 \cdot \frac{1}{n^{3/2}}}{1+n^3 \cdot \frac{1}{n^3}} = \frac{\sqrt{n}}{2}$$

$$\therefore M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\frac{n^2 x}{1+n^3x^2}\right| = \frac{\sqrt{n}}{2}$$

Which does not tend to zero as n $\to\infty$

$\therefore$ By M_n – test the sequence $< f_n >$ does not converge uniformly on [0, 1].

10. Let $\{f_n\}$ be sequence of real valued functions defined on a set E. Then $\{f_n\}$ converges uniformly on E iff for every ε> 0 ∃a Positive integer *t* st.

$$[f_n(x) - f_m(x)] < \varepsilon \;\; \forall n, m \geq t, \;\; x \in E.$$

(September 2012, 2009, April 2012)

Sol. Let $f_n \to f$ converge uniformly on [a, b]. Then ∀ε> 0

∃ an integer t = t (ε) s.t.

$$n \geq t, x \in [a,b] \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2} \qquad (1)$$

Therefore, for every $x \in [a,b] m, n \geq t$

$$\Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ and } |f_m(x) - f(x)| < \frac{\varepsilon}{2} \qquad (2)$$

We have

$$|f_n(x) - f_m(x)| = |[f_n(x) - f(x)] + [f(x) - f_m(x)]|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{[Using (2)]}$$

For every x ∈ [a, b] m, n ≥t

Conversely. Suppose x ∈ [a, b] and m, n ≥t

$$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon \qquad (1)$$

Then, for each, x ∈ [a, b]

$<f_n(x)>$ is a Cauchy sequence of set of real numbers ℝ.

Since, ℝ is complete

∃ a real number 'y' s.t $\lim_{n\to\infty} f_n(x) = y$

Define a function *f*: E → ℝ by *f*(*x*) = y ∀ x ∈ [a, b]

$$f(x) = y = \lim_{n\to\infty} f_n(x)$$

$$\Rightarrow f_n \xrightarrow{p.w.} f \text{ on [a, b]}$$

We claim $f_n \to f$ converge uniformly on [a, b].

In equation (i), keeping n fixed and m $\to\infty$

We get

$$|f_n(x) - f(x)| < \varepsilon \forall n \geq t \text{ and } x \in [a,b]$$

$\Rightarrow f_n$ converge uniformly to *f* on [a, b]

11. Prove that:

$$\frac{x}{1+x^2} + \left\{\frac{2^2 x}{1+2^3x^2} - \frac{x}{1+x^2}\right\} + \left\{\frac{3^2 x}{1+3^3x^2} - \frac{2^2 x}{1+2^3x^2}\right\} + \ldots\ldots$$ **does not converges uniformly on [0, 1].**

(April 2012)

Sol. We have $u_1(x) = \frac{x}{1+x^2}$

$$u_2(x) = \frac{2^2 x}{1+2^3x^2} - \frac{x}{1+x^2}$$

$$u_3(x) = \frac{3^2 x}{1+3^3x^2} - \frac{2^2 x}{1+2^3x^2}$$

....

$$u_n(x) = \frac{n^2 x}{1+n^3x^2} - \frac{(n-1)^2 x}{1+(n-1)^3 x^2}$$

Adding $f_n(x) = \frac{n^2 x}{1+n^3x^2}$

Hence $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{\frac{1}{n^2} + nx^2} = 0 \;\; \forall x \in [0,1]$

Now $M_n = \text{Sup}[|f_n(x) - f(x)| : x \in [0,1]]$

$$M_n = \text{Sup}\left[\frac{n^2 x}{1+n^3x^2} : x \in [0,1]\right]$$

$$\geq \frac{n^2 \frac{1}{n^{3/2}}}{1+n^3 \frac{1}{n^3}} = \frac{\sqrt{n}}{2} \to \infty \text{ as } n \to \infty \qquad \left[taking\ x = \frac{1}{n^{\frac{3}{2}}} \in [0,1] \right]$$

Since M_n does not tend to zero as $n \to \infty$, the series is non-uniformly convergent on [0, 1] by M_n-Test. Here 0 is a point of non-uniform convergence.

12. Test the sequence $\{f_n(x)\}$ for uniform convergence on [0, 1], where: (April 2012)

(i) $f_n(x) = \dfrac{n^2 x}{1+n^4 x^2}$ **(ii)** $f_n(x) = \dfrac{nx}{1+n^3 x^2}$

Sol. (i) $f_n(x) = \dfrac{n^2 x}{1+n^4 x^2} \to 0$ as $n \to \infty$

$\therefore f(x) = 0 \quad x \neq 0 \quad \forall x \in [0,1]$

$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$

$= \sup\left\{\dfrac{n^2 x}{1+n^4 x^2} : x \in [0,1]\right\}$

$$\geq \frac{n^2 . \frac{1}{n^2}}{1+n^4 \frac{1}{n^4}} = \frac{1}{2} \neq 0 \qquad \left[\text{Taking } x = \frac{1}{n^2} \in [0,1]\right]$$

$\Rightarrow M_n$ does not tend to zero as $n \to \infty$ in [0, 1]

$\Rightarrow$ The series in not uniformly convergence in [0, 1] by M_n-test.

(ii) Let $f_n(x) = \dfrac{nx}{1+n^3 x^2}$

then, $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{nx}{1+n^3 x^2} = \lim\limits_{n\to\infty} \dfrac{\frac{n}{x^2}}{\frac{1}{n^3} + x^2} = \dfrac{0}{x^2} = 0$

$\Rightarrow f(x) = 0\ \forall$ x

Now $|f_n(x) - f(x)| = \left|\dfrac{nx}{1+n^3x^2} - 0\right| = \left|\dfrac{nx}{1+n^3x^2}\right|$

Let $y = \dfrac{nx}{1+n^3x^2}$

$$\frac{dy}{dx} = \frac{(1+n^3x^2)(n) - (nx)(2n^3x)}{(1+n^3x^2)^2} = \frac{n - n^4x^2}{(1+n^3x^2)^2}$$

For max. or min. $\dfrac{dy}{dx} = 0$

$\Rightarrow n - n^4x^2 = 0 \quad \Rightarrow x^2 = \dfrac{1}{n^3} \qquad \Rightarrow x = \dfrac{1}{n^{3/2}}$

$$\frac{d^2y}{dx^2} = \frac{(1+n^3x^2)^2(-2n^4x) - (n-n^4x^2)2(1+n^3x^2)(2n^3x)}{(1+n^3x^2)^4}$$

$$= \frac{-2n^3x(3n - n^4x^2)}{(1+n^3x^2)^3} \qquad \text{(On simplification)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n^{3/2}}} = \frac{-2n^3 . \frac{1}{n^{3/2}}\left(3n - n^4 . \frac{1}{n^3}\right)}{\left(1+n^3 . \frac{1}{n^3}\right)^3} = \frac{-4n^{5/2}}{8} = \frac{-n^{5/2}}{2} < 0$$

Which shows y is maximum when x = $\dfrac{1}{n^{3/2}}$.

$y = \dfrac{nx}{1+n^3x^2}$

$$\therefore\ y_{max} = \frac{n \frac{1}{n^{3/2}}}{1+n^3 . \frac{1}{n^3}} = \frac{1}{2\sqrt{n}},$$

$$\therefore M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\frac{nx}{1+n^3x^2}\right| = \frac{1}{2\sqrt{n}} \to 0, \text{as } n \to \infty$$

$\therefore$By M_n – test the sequence $< f_n >$ converge uniformly on [0, 1].

13. Let $\{f_n\}$ be sequence of function such that on [a, b]

$\lim\limits_{n\to\infty} f_n(x) = f(x)$ and $M_n = Sup|f_n(x) - f(x)|$ on [a, b]

Show $f_n \to f$ uniformly iff $M_n \to 0$ as $n \to \infty$. (September 2011)

Sol. Necessary Condition: Let the sequence $\{f_n\}$ converges uniformly to f on X = [a, b]. Then for an ε> 0, there exists a +ve integer m, independent of x, such that

$\forall\ n \geq m, x \in X \Rightarrow |f_n(x) - f(x)| < \varepsilon$ (1)

Since M_n is the supermum of $|f_n(x) - f(x)|$ for varying x, it follows from (1) that

$\forall\ n \geq m \Rightarrow M_n < \varepsilon$

Hence $M_n \to 0 \qquad \Rightarrow n \to \infty$

Sufficient Condition: Let $M_n \to 0$ as $n \to \infty$. For given ε> 0 ∃ a +ve integer m such that

$\forall\ n \geq m \Rightarrow M_n < \varepsilon$

But M_n is the supermum of $|f_n(x) - f(x)|$ for varying x

Hence $|f_n(x) - f(x)| \leq M_n < \varepsilon\ \forall x \in$ X and $\forall n \geq m$

$\therefore \{f_n\}$ converges uniformly to f on X.

14. Show that the series $\sum \frac{1}{n^2 + n^4x^2}$ is uniformly convergent for all x but cannot be differentiated term by term. **(September 2011)**

Sol. Given $u_n(x) = \frac{1}{n^2 + n^4x^2}$

$$u_n'(x) = \frac{-2x}{(n^2 + n^4x^2)^2}$$

Now $u_n'(x)$ is maximum when $\frac{d}{dx}(u_n'(x)) = 0$

i.e. $(n^2 + n^4x^2)^2(-2) - (-2x).2(n^2 + n^4x^2)(2xn^4) = 0$

$\Rightarrow (n^2 + n^4x^2)[-2n^2 - 2n^4x^2 + 8n^4x^2] = 0$

$\Rightarrow (n^2 + n^4x^2)[-2n^2 + 6n^4x^2] = 0$

$\Rightarrow [-2n^2 + 6n^4x^2] = 0 \Rightarrow -n^2 + 3n^4x^2 = 0$

$\Rightarrow x = \frac{1}{\sqrt{3n}}$

$$\therefore \text{Max. } |u_n'(x)| = \left| \frac{-2.\frac{1}{\sqrt{3n}}}{\left(n^2 + n^4\left(\frac{1}{3n^2}\right)\right)^2} \right| = \left| \frac{\frac{-2}{\sqrt{3n}}}{\left(\frac{4}{3}n^2\right)^2} \right| = \frac{2}{\sqrt{3n}} \cdot \frac{9}{16n^4} = \frac{9}{8\sqrt{3n^5}}$$

Then $|u_n'(x)| < \frac{1}{n^5} \qquad \forall x$

But $\sum \frac{1}{n^5}$ is convergent

Hence by Weierstrass M-test the series $\sum u_n'(x)$ is uniformly convergent. For all x

Hence $f'(x) = \sum_{n=1}^{\infty} u_n'(x) = \sum_{n=1}^{\infty} \frac{-2x}{(n^2 + n^4x^2)^2}$

$\therefore$ the series $\sum u_n(x)$ can be differentiated term by team

Convergence of $\sum u_n$.

We have $u_n = \frac{1}{n^2 + n^4x^2}$

$$\therefore |u_n| = \left| \frac{1}{n^2 + n^4x^2} \right| = \left| \frac{1}{n^2(1 + n^2x^2)} \right| = \left| \frac{1}{n^2} \right| \left| \frac{1}{1 + n^2x^2} \right|$$

$$\leq \frac{1}{n^2} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

By Weierstrass M-test the given series is uniformly convergent.

15. Let $f_n(x) = \frac{1}{1 + nx}$, show that series $\sum f_n(x)$ is not uniformly convergent on [0, 1] but can be integrated term by term. **(September 2011, April 2009)**

Sol. Here

$$f_n(x) = \frac{1}{1 + nx}$$

$$\therefore f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{1 + nx} = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & x = 0 \end{cases}$$

For $0 < x \leq 1$ and for given ε> 0, we have

$$|f_n(x) - f(x)| = \left| \frac{1}{1 + nx} - 0 \right| = \frac{1}{1 + nx} < \varepsilon$$

If $1 + nx > \frac{1}{\varepsilon}$

If $nx > \frac{1}{\varepsilon} - 1$

If $n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$

$\Rightarrow n \to \infty$ if $x \to 0$

So that x = 0 is a point of non-uniform convergence of given series.

Now $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

and $\int_0^1 f_n(x)\,dx = \int_0^1 \frac{1}{1+nx}\,dx$

$= \left[\frac{\log(1+nx)}{n}\right]_0^1 = \frac{\log(1+n)}{n}$

Now $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{\log(1+n)}{n}$

$= \lim_{n\to\infty} \frac{\frac{1}{1+n}}{1} = \frac{0}{1} = 0$ $\left[\frac{\infty}{\infty} \text{ form } \therefore \text{ using L'Hospital Rule}\right]$

Hence $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)dx$

The series is integrable term by term on [0, 1], although x = 0 is point of non-uniform convergence of the series.

16. Show series $\sum \frac{x}{n^p + x^2 n^q}$ converge uniformly over any finite interval [a, b] for

(i) p > 1, q ≥0 **(ii) 0 < p ≤ 1, p + q > 2.** **(April 2011)**

Sol. (i) When p > 1, q ≥ 0

Here $f_n(x) = \frac{x}{n^p + n^q x^2}$

Since $x^2 \geq 0 \quad \forall x \in [a,b]$

$\Rightarrow n^q x^2 \geq 0$ $\Rightarrow n^p + n^q x^2 \geq n^p$

$\Rightarrow \frac{1}{n^p + n^q x^2} \leq \frac{1}{n^p}$ $\Rightarrow \frac{x}{n^p + n^q x^2} \leq \frac{x}{n^p}$ (since x ≥ 0)

$\therefore |f_n(x)| = \left|\frac{x}{n^p + n^q x^2}\right| \leq \left|\frac{x}{n^p}\right| \leq \frac{\alpha}{n^p} = M_n$

Where $\alpha \geq \max \{|a|, |b|\}$

Since $\sum M_n = \sum \frac{\alpha}{n^p}$ is convergent for p > 1

Therefore, by Weirestrass M-test the given series is uniformly convergent for all x ∈ [a, b]

(ii) when 0 < p ≤ 1, p + q > 2

Since $f_n(x) = \frac{x}{n^p + n^q x^2}$

$\therefore \frac{df_n(x)}{dx} = \frac{(n^p + n^q x^2).1 - x(2n^q x)}{(n^p + n^q x^2)^2} = \frac{(n^p - n^q x^2)}{(n^p + n^q x^2)^2}$

To find max. or min. of $f_n(x)$

$\frac{df_n(x)}{dx} = 0$

$\Rightarrow n^p - n^q x^2 = 0 \Rightarrow x^2 = n^{p-q} \quad \Rightarrow x = n^{\frac{p-q}{2}}$

Also $\frac{d^2 f_n(x)}{dx^2} = \frac{-2n^q x(3n^p - n^q x^2)}{(n^p + n^q x^2)^3}$ (On simplification)

Now $\left.\frac{d^2 f_n(x)}{dx^2}\right]_{x=n^{\frac{p-q}{2}}} = \frac{-2n^q n^{\frac{p-q}{2}}(3n^p - n^q n^{p-q})}{(n^p + n^q n^{p-q})^3}$

$= \frac{-2n^q n^{\frac{p-q}{2}}(3n^p - n^p)}{(n^p + n^p)^3} = \frac{-2n^{\frac{p+q}{2}}(2n^p)}{(2n^p)^3} = -\frac{1}{2} n^{\frac{q-3p}{2}} < 0$

Which shows $f_n(x)$ is maximum at $x = n^{\frac{p-q}{2}}$ and Maximum value of

$f_n(x) = f_n(x)]_{x=n^{\frac{p-q}{2}}} = \frac{n^{\frac{p-q}{2}}}{n^p + n^p} = \frac{1}{2n^{\frac{p+q}{2}}}$

$\Rightarrow |f_n(x)| = \left|\frac{x}{n^p + n^q x^2}\right| \leq \frac{1}{2n^{\frac{p+q}{2}}} < \frac{1}{n^{\frac{p+q}{2}}}$

Since $\sum M_n = \sum \frac{1}{n^{\frac{p+q}{2}}}$ is convergent if $\frac{p+q}{2} > 1$ i.e. p + q > 2

Therefore by weierestrass M-test the given series is convergent for all x ∈ [a, b] if p + q > 2.

17. Show that sequence of functions $f_n(x) = e^{-nx}$ is uniformly convergent in any interval [a, b] but only pointwise convergent on [0, b]. (April 2011)

Sol. Given $f_n(x) = e^{-nx}$

Sum function $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} = 0$ when x > 0

At x = 0

$f_n(0) = e^{-n(0)} = 1$

$f(0) = \lim_{n\to\infty} f_n(0) = 1$

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$$

Let ε> 0, however small be given s.t. for x > 0

$|f_n(x) - f(x)| < \varepsilon$

$|e^{-nx} - 0| < \varepsilon$

$\Rightarrow e^{-nx} < \varepsilon$

$\Rightarrow e^{nx} > \frac{1}{\varepsilon}$

$\Rightarrow nx > \log\left(\frac{1}{\varepsilon}\right)$

$\Rightarrow n > \frac{\log\left(\frac{1}{\varepsilon}\right)}{x}$

Clearly $\frac{\log\left(\frac{1}{\varepsilon}\right)}{x}$ decreases as x increases

$\therefore$ max value of $\frac{\log\left(\frac{1}{\varepsilon}\right)}{x}$ is $\frac{\log\left(\frac{1}{\varepsilon}\right)}{a}$ when x ∈ [a, b] and a > 0

Let M be an integer such that $M \geq \frac{\log\left(\frac{1}{\varepsilon}\right)}{a}$

$\therefore$ for given ε> 0 however small ∃ an integer M such that
∀ x∈ [a, b], a > 0

$|f_n(x) - f(x)| < \varepsilon \ \forall n \geq M$

$\therefore$ Sequence $< f_n(x) >$ converges uniformly on [a, b], a > 0

However $\lim_{x\to 0} \frac{\log\left(\frac{1}{\varepsilon}\right)}{x} = \infty$

$\therefore$ No such integer M exists such that

$|f_n(x) - f(x)| < \varepsilon \qquad \forall n \geq M$

Hence the sequence $< f_n(x) >$ is not uniformly convergent ∀x∈ [0, b] but it is point wise convergent in [0, b]

18. Show that series $1 - x + x^2 - x^3 + \ldots\ldots + \infty$ can be integrated term by term on [0, 1] even though the series is not uniformly convergent on [0, 1]. (April 2011)

Sol. When x = 1, The given series is
1 – 1 + 1 – 1 + ... Oscillate
For 0 ≤ x < 1

$$1 - x + x^2 - x^3 + \ldots\ldots = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

Thus, the series is not uniformly convergent on [0 1]
x = 1 is the point of non-uniform convergence of the series.

Now, $\int_0^1 1dx - \int_0^1 xdx + \int_0^1 x^2 dx - \ldots\ldots\ldots\ldots$

$$= [x]_0^1 - \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1 \ldots\ldots\ldots = 1 - \frac{1}{2} + \frac{1}{3} - \ldots\ldots = \log 2$$

Also $\int_0^1 \frac{1}{1+x} dx = \left[\log(1+x)\right]_0^1 = \log 2$

Hence term by term integration is possible.

19. If series $\sum f_n$ uniformly converges on [a, b] and each f_n is continuous on [a, b] then series $\sum\left(\int_a^x f_n dt\right)$ Converges uniformly to $\int_a^x f dt$ on [a, b]. (April 2011)

Sol. Since $\sum f_n$ is uniformly convergent to f on [a, b] and each f_n is continuous on [a, b]

$\therefore$ the sum function f is continuous and hence integrable on [a, b]

Since all the function f_n are continuous

$\therefore$ the sum of a finite number of function $\sum_{r=1}^{n} f_r$ is also continuous and integrable on [a, b] and $\sum_{r=1}^{n}\int_a^x f_r dt = \int_a^x \sum_{r=1}^{n} f_r dt$

Also given $\sum f_n$ converges to f

∴ By def. for given ε> 0, however small we can find a +ve integer N s.t. ∀x∈ [a, b]

$$\left|f-\sum_{r=1}^{n} f_r\right| < \frac{\varepsilon}{b-a} \qquad \forall n \geq N$$

Consider $\left|\int_a^x f dt - \sum_{r=1}^{n}\int_a^x f_r dt\right| = \left|\int_a^x\left(f-\sum_{r=1}^{n} f_r\right)dt\right|$

$$\leq \int_a^x \left|f-\sum_{r=1}^{n} f_r\right| dt < \frac{\varepsilon}{b-a}\int_a^x dt = \frac{\varepsilon}{b-a}(x-a) \leq \varepsilon \qquad \because x\in[a,b]$$

$$\Rightarrow \left|\int_a^x f dt - \sum_{r=1}^{n}\int_a^x f_r dt\right| < \varepsilon$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right) \text{ converges uniformly to } \int_a^x f dt$$

$$\therefore \int_a^x f dt = \sum_{n=1}^{\infty}\int_a^x f_n dt \qquad \forall x\in [a, b]$$

20. What is difference between pointwise and uniform convergence of sequence of functions on an interval. Show sequence $f_n(x)=\dfrac{nx}{1+n^2x^2}$ is pointwise convergent but not uniformly convergent on any interval containing zero.

(September 2010)

Sol. POINT –WISE CONVERGENCE

Suppose $< f_n >$, n = 1, 2, 3, ... is a sequence of a function defined on [a, b], then the sequence $< f_n >$ converges point-wise to a real-valued function f defined on [a, b],I written as

$f_n \xrightarrow{p.w.} f$ on [a, b]

If $\forall x\in[a,b], \lim_{n\to\infty} f_n(x)=f(x)$ i.e. $\forall x\in[a,b]$ and given

any ε> 0 however small ∃ an integer N (dependent on x and ε) such that

$n\geq N \quad \Rightarrow \quad |f_n(x)-f(x)|<\varepsilon$

UNIFORM CONVERGENCE

The sequence $< f_n >$, n = 1, 2, 3 ...., converge uniformly on [a, b] to a function *f* if for every ε> 0.

∃ an integer N (independent of x and depend on ε) such that

$\Rightarrow |f_n(x)-f(x)|<\varepsilon \;\; \forall n\geq N$

- It is clear from the definition that Uniform convergence ⇒ point-wise convergence and Uniform limit = point-wise limit.

Here

$$f_n(x)=\frac{nx}{1+n^2x^2}$$

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{nx}{1+n^2x^2}=\lim_{n\to\infty}\frac{\frac{x}{n}}{\frac{1}{n^2}+x^2}=\frac{0}{0+x^2}=0$$

$$\Rightarrow f(x)=0 \quad \forall x\in \mathbb{R}$$

Now $|f_n(x)-f(x)|=\left|\dfrac{nx}{1+n^2x^2}-0\right|=\left|\dfrac{nx}{1+n^2x^2}\right|$

Let $y=\dfrac{nx}{1+n^2x^2}$

$$\frac{dy}{dx}=\frac{(1+n^2x^2)(n)-(nx)(2n^2x)}{(1+n^2x^2)^2}=\frac{n-n^3x^2}{(1+n^2x^2)^2}$$

For max or min $\dfrac{dy}{dx}=0$

$$\Rightarrow n-n^3x^2=0 \quad \Rightarrow x^2=\frac{1}{n^2}\Rightarrow x=\frac{1}{n}$$

Now $\dfrac{dy}{dx}=\dfrac{n-n^3x^2}{(1+n^2x^2)^2}$

$$\therefore \frac{d^2y}{dx^2}=\frac{(1+n^2x^2)^2(-2n^3x)-(n-n^3x^2)2(1+n^2x^2)(2n^2x)}{(1+n^2x^2)^4}=\frac{-2n^2x(3n^3x^2-n)}{(1+n^2x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n}}=\frac{-2n^2\left(\frac{1}{n}\right)\left(3n^3\frac{1}{n^2}-n\right)}{\left(1+n^2\frac{1}{n^2}\right)^3}=\frac{-2n(3n-n)}{(1+1)^3}=\frac{-4n^2}{8}=\frac{-n^2}{2}<0$$

$\Rightarrow$ y is maximum, when $x = \frac{1}{n}$

$$y_{max} = \frac{n.\frac{1}{n}}{1+n^2.\frac{1}{n^2}} = \frac{1}{1+1} = \frac{1}{2}$$

Thus $M_n = \underset{x\in[a,b]}{Sup}|f_n(x) - f(x)| = \underset{x\in[a,b]}{Sup}\left|\frac{nx}{1+n^2x^2} - 0\right| = \underset{x\in[a,b]}{Sup}\left|\frac{nx}{1+n^2x^2}\right| = \frac{1}{2}$

Thus, M_n does not tend to zero as $n \to \infty$

$\therefore$ By M_n-test sequence $< f_n >$ is not uniformly convergent in any interval containing zero.

21. Show that the series $1 + \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} + \frac{e^{-6x}}{6^2-1} + \ldots\ldots\ldots\ldots$ is uniformly convergent for x non-negative. **(September 2010)**

Sol. Let $u_n(x) = \frac{e^{-2nx}}{(2n)^2 - 1} = \frac{e^{-2nx}}{4n^2-1}$

$\therefore |u_n(x)| = \left|\frac{e^{-2nx}}{4n^2-1}\right| \le \frac{1}{4n^2} \le \frac{1}{n^2} \quad \forall n \ge 0$

and $\sum \frac{1}{n^2}$ cgs

$\therefore$ By W.M. Test, the given series is uniform cgs $\forall n \ge 0$.

22. Test for convergence of the series: **(April 2010)**

$$\frac{1}{(1+x)^3} + \frac{2}{(2+x)^3} + \frac{3}{(3+x)^3} \ldots\ldots\ldots \text{ for } x \ge 0.$$

Sol. The given series is $\sum_{n=1}^{\infty} \frac{n}{(n+x)^3}$

Here $f_n(x) = \frac{n}{(n+x)^3}$

Now $|f_n(x)| = \left|\frac{n}{(n+x)^3}\right| = \frac{n}{(n+x)^3} \le \frac{n}{n^3} = \frac{1}{n^2} \quad \forall \quad x \ge 0$

$\Rightarrow M_n = \frac{1}{n^2}$, By p-test $\sum M_n = \sum \frac{1}{n^2}$ is convergent. Hence, by weierstrass M-test, the given series is uniformly convergent $\forall x \ge 0$.

23. Show the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ can be differentiated term by term and find its derivative. **(April 2010, September 2009)**

Sol. Let $f(x) = \sum_1^{\infty} \frac{\sin nx}{n^3}$

and $u_n(x) = \frac{\sin nx}{n^3}$

$u_n'(x) = \frac{\cos nx}{n^2}$

$\therefore \sum_1^{\infty} u_n'(x) = \sum_1^{\infty} \frac{\cos nx}{n^2}$

But $\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}$ for all x and $\sum \frac{1}{n^2}$ is cgt.

By W.M. Test $\sum u_n'(x)$ is uniformly convergent for all x and therefore $\sum u_n(x)$ can be differentiated term by term

$\therefore f'(x) = \sum_1^{\infty} u_n'(x) = \sum_1^{\infty} \frac{\cos nx}{n^2}$

24. Prove that a series of functions $\sum f_n$ will converge uniformly and absolutely on [a, b] if there exists a convergent series $\sum M_n$ of +ve number such that for all $x \in [a,b]$, $|f_n(x)| \le M_n$ for all n. **(April 2009)**

Sol. Since $\sum M_n$ is convergent.

$\therefore$ for $\varepsilon > 0 \ \exists \ m \in \mathbb{N}$ s.t.

$M_{n+1} + M_{n+2} + \ldots\ldots + M_{n+p} < \varepsilon \ \forall \ n \ge m, p \in \mathbb{N}$ (1)

Also $|f_n(x)| \le M_n$ (2)

$\therefore$ from (1) and (2)

$|f_{n+1}(x) + f_{n+2}(x) + \ldots.. + f_{n+p}(x)|$

$\le |f_{n+1}(x)| + |f_{n+2}(x)| + \ldots.. + |f_{n+p}(x)|$

$\le M_{n+1} + M_{n+2} + \ldots\ldots + M_{n+p} < \varepsilon$

$\forall n \ge m, \ \forall x \in X = [a, b]$

Hence $\sum f_n(x)$ cgs uniformly and absolutely on X = [a, b].

25. **Show that the sequence $\{f_n\}$ where $f_n(x)=\dfrac{x}{n(1+nx^2)}$ is uniformly convergent for all x.** **(April 2009)**

Sol. Here $f_n(x)=\dfrac{x}{n(1+nx^2)}$

$$\therefore |f_n(x)|=\left|\frac{x}{n(1+nx^2)}\right|$$

Now $\dfrac{d}{dx}(f_n(x))=\dfrac{(1+nx^2)-x.2nx}{n(1+nx^2)^2}$

$$=\frac{1-nx^2}{n(1+nx^2)^2}$$

For max or min $\dfrac{d}{dx}(f_n(x))=0$

$\Rightarrow 1-nx^2=0$

$\Rightarrow nx^2=1 \qquad \Rightarrow x=\pm\dfrac{1}{\sqrt{n}}$

When $x<\dfrac{1}{\sqrt{n}}, \dfrac{d}{dx}(f_n(x))>0$

and when $x>\dfrac{1}{\sqrt{n}}, \dfrac{d}{dx}(f_n(x))<0$.

$\Rightarrow \dfrac{d}{dx}(f_n(x))$ change sign from +ve to –ve

$\Rightarrow f_n(x)$ is Max. at $x=\dfrac{1}{\sqrt{n}}$

So that Max. $|f_n(x)|=\left|\dfrac{\frac{1}{\sqrt{n}}}{n\left(1+\frac{n}{n}\right)}\right|=\left|\dfrac{1}{2n^{3/2}}\right|$

$\Rightarrow |f_n(x)|\le\dfrac{1}{2}\dfrac{1}{n^{3/2}}$ and $\sum\dfrac{1}{n^{3/2}}$ is convergent.

$\therefore$ By W.M. Test the given series is uniformlycgt $\forall x$.

26. **Given the series $\sum f_n'(x)$ for which $f_n(x)=\dfrac{1}{2n^2}\left[\log(1+n^4x^2)\right]$. Show that the series $\sum f_n'(x)$ does not converge uniformly, but the given series is differentiable term by term.** **(April 2009)**

Sol. Let $f_n(x)=\dfrac{1}{2n^2}\log(1+n^4x^2)$, $\qquad 0\le x\le 1$

$\Rightarrow \lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty}\dfrac{1}{2n^2}\log(1+n^4x^2)$

$=0$

Also consider $g(x)=\lim\limits_{n\to\infty} f_n'(x)=\lim\limits_{n\to\infty}\dfrac{1}{2n^2}\cdot\dfrac{2n^4x}{1+n^4x^2}$

$=\lim\limits_{n\to\infty}\dfrac{xn^2}{1+n^4x^2}=0 \qquad (0\le x\le 1)$

Thus $\qquad f_n'(x)=g(x) \qquad (0\le x\le 1)$

However, $\left|f_n'(x)-g(x)\right|=\dfrac{xn^2}{1+n^4x^2} \qquad$ **(1)**

Taking $x=\dfrac{1}{n^2}$, we get maximum value of (1) is $\dfrac{1}{2}$

So that $M_n=\sup\left|f_n'(x)-g(x)\right|=\dfrac{1}{2}$

So, M_n does not tends to zero as $n\to\infty$

Hence by M_n – test

$\sum f_n'(x)$ does not converge uniformly to g(x) on [0, 1]

Also $\qquad f_n'(x)=g(x) \qquad \forall x\in[0,1]$

We know $f(x)=\sum\limits_{n=1}^{\infty} f_n(x)$

$\Rightarrow \dfrac{d}{dx}\sum\limits_{n=1}^{\infty} f_n(x)=\sum\limits_{n=1}^{\infty}\dfrac{d}{dx}f_n(x) \qquad \forall x\in[0,1]$

$\Rightarrow$ The $\sum f_n(x)$ is differentiable term by term

8

POWER SERIES

1. **Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(\lfloor n)^2}{\lfloor 2n}x^{2n}$.** **(September 2013)**

Sol. The given series is $\sum_{n=1}^{\infty}\frac{(\lfloor n)^2}{\lfloor 2n}x^{2n}$

Take $x^2 = t$, the series becomes $\sum_{n=1}^{\infty}\frac{(\lfloor n)^2}{\lfloor 2n}t^n$

$$a_n = \frac{(\lfloor n)^2}{\lfloor 2n},\ a_{n+1} = \frac{(\lfloor n+1)^2}{\lfloor 2n+2}$$

$$\frac{1}{R} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(\lfloor n+1)^2}{\lfloor 2n+2}\times\frac{\lfloor 2n}{(\lfloor n)^2}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{n+1}{2(2n+1)}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{4\left(1+\frac{1}{2n}\right)} = \frac{1}{4}$$

$\therefore$ Radius of convergence of series in $t = R = 4$

$\therefore$ the given series is convergent for $|t| < 4$

$\Rightarrow -4 < t < 4$

$\Rightarrow -4 < x^2 < 4$

$\Rightarrow 0 < x^2 < 4$ $\qquad [\because x \text{ is real}]$

$\Rightarrow |x| < 4$

$\Rightarrow -2 < x < 2$

$\Rightarrow x \in (-2, 2)$ i.e. interval of convergence (–2, 2)

2. **Prove that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots\ldots$ for $-1 \le x \le 1$.** **(September 2013)**

Sol. We have $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots..$ $\qquad$ (A)

$= \sum_{n=0}^{\infty}(-1)^n .x^{2n}$

$= \sum_{n=0}^{\infty}(-1)^n .t^n$ $\qquad$ [Taking $x^2 = t$]

Here $a_n = (-1)^n$

$a_{n+1} = (-1)^{n+1}$

$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = 1$

$\therefore$ for interval of convergence

$\Rightarrow |t| < 1$

$\Rightarrow |x^2| < 1$

$\Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$

Therefore the series on the R.H.S. is a power series with radius of convergence unity and converges absolutely for $-1 < x < 1$

As such it converges uniformly in $(-\lambda, \lambda)$ where $|\lambda| < 1$

The series on R.H.S. does not converge for $x = \pm 1$

($\because$ it oscillates for these values)

The integrating series will have the same characteristics.

Thus integrating both sides w.r.t. x

We get

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.. + c$$

When $x = 0,\ c = 0$

$\therefore\ \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \text{ for } -1 < x < 1$

Because for $x = \pm 1$ the power series on R.H.S. become

$\pm\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right) = \pm(-1)^{n-1}\cdot\frac{1}{2n-1}$

Which is an alternating series

$\therefore$ By Leibnitz's test the power series on the R.H.S. is convergent for $x = \pm 1$ also.

Therefore it converges in [–1, 1] and hence converges uniformly for $x \in [-1, 1]$

Thus $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^4}{5} - \frac{x^7}{7} + \ldots\ldots \text{ for } -1 \le x \le 1$

3. Show that $\int_0^1 \frac{\tan^{-1}x}{x}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$. **(April 2013)**

Sol. We have $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$ **(A)**

$= \sum_{n=0}^{\infty}(-1)^n \cdot x^{2n}$

$= \sum_{n=0}^{\infty}(-1)^n \cdot t^n$ $\{\text{Taking } x^2 = t\}$

Here $a_n = (-1)^n$

$a_{n+1} = (-1)^{n+1}$

$\underset{n\to\infty}{\text{Lt}}\left|\frac{a_{n+1}}{a_n}\right| = 1$

$\therefore$ for interval of convergence

$\Rightarrow |t| < 1$

$\Rightarrow |x^2| < 1$

$\Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$

Therefore the series on the R.H.S. is a power series with radius of convergence unity and converges absolutely for $-1 < x < 1$

As such it converges uniformly in $(-\lambda, \lambda)$ where $|\lambda| < 1$

The series on R.H.S. does not converge for $x = \pm 1$

($\because$ it oscillates for these values)

The integrated series will have the same characteristics.

Thus integrating both sides w.r.t. x

We get

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + c$

When $x = 0, c = 0$

$\therefore\ \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \text{ for } -1 < x < 1$

Because for $x = \pm 1$ the power series on R.H.S. become

$\pm\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right) = \pm(-1)^{n-1}\cdot\frac{1}{2n-1}$

Which is an alternating series

$\therefore$ By Leibnitz's test the power series on the R.H.S. is convergent for $x = \pm 1$ also. Therefore it converges in [–1, 1] and hence converges uniformly for $x \in$ [–1, 1]

Thus $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \text{ for } -1 \le x \le 1$ **(B)**

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \text{ for } |x| \le 1$

dividing both sides by x

$\frac{\tan^{-1}x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots\ldots\ldots$ integrating both sides w.r.t. x from 0 to 1

$\int_0^1 \frac{\tan^{-1}x}{x}dx = \left[x - \frac{x^3}{(3)^2} + \frac{x^5}{(5)^2} - \frac{x^7}{(7)^2} + \ldots\ldots\right]_0^1$

$= 1 - \frac{1}{(3)^2} + \frac{1}{(5)^2} - \frac{1}{(7)^2} + \ldots\ldots$

$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$.

4. Find the interval of convergence of power series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n^2}$. **(April 2013)**

Sol. The power series is $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n^2}$

Take $x + 2 = t$

The given series become $\sum_{n=1}^{\infty}\frac{t^n}{n^2}$

Here $a_n = \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1$$

$\therefore$ Radius of convergence R = 1

$\Rightarrow$ The series is convergent for $|t| < 1$

$\Rightarrow -1 < t < 1$

$\Rightarrow -1 < x + 2 < 1$

$\Rightarrow -1 - 2 < x < 1 - 2$

$\Rightarrow -3 < x < -1$

$\Rightarrow$ Interval of convergence is $(-3, -1)$

5. **Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty}\frac{|\underline{2n}}{(|\underline{n})^2}x^n$.**

(September 2012, April 2010)

Sol. Here $a_n = \frac{|\underline{2n}}{(|\underline{n})^2}$

$$a_{n+1} = \frac{|\underline{2n+2}}{(|\underline{n+1})^2}$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\left|\frac{a_n}{a_{n+1}}\right| = \underset{n\to\infty}{\text{Lt}}\frac{(|\underline{n+1})^2}{(|\underline{2n+2})}\times\frac{|\underline{2n}}{(|\underline{n})^2}$$

$$= \underset{n\to\infty}{\text{Lt}}\left[\frac{|\underline{n+1}}{|\underline{n}}\right]^2\times\frac{|\underline{2n}}{(2n+2)(2n+1)|\underline{2n}}$$

$$= \underset{n\to\infty}{\text{Lt}}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \underset{n\to\infty}{\text{Lt}}\frac{n+1}{2(2n+1)}$$

$$= \underset{n\to\infty}{\text{Lt}}\frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)} = \frac{1}{4}$$

$\therefore$ Radius of convergence = R = $\frac{1}{4}$ = 0.25

$\therefore$ The given series converges for

$|x| < 0.25$

$\Rightarrow -0.25 < x < 0.25$

$\Rightarrow x \in (-0.25,\ 0.25)$

$\therefore$ interval of convergence is $(-0.25,\ 0.25)$

6. **Show that $\int_0^x \log(1+x)\,dx = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{x^{n+1}}{n(n+1)}$ for $|x| < 1$.**

Does the result hold for $x = \pm 1$? **(September 2012)**

Sol. We know

$$\log(1-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\text{..........} \text{ for } -1 < x \le 1$$

Integrating both sides w.r.t. x from 0 to x.

$$\int_0^x \log(1+x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} - \frac{x^5}{4.5} + \text{......}\right]_0^x$$

$$= \frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} - \frac{x^5}{4.5} + \text{..........} \quad (1)$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{(n+1)}}{n(n+1)} \quad \text{for } |x| < 1$$

For x = 1, the series on the R.H.S. becomes $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n(n+1)}$

Which is an alternating series and is convergent by Leibnitz's test.

The result holds for x = 1

For x = –1, the series on the R.H.S. of (1) becomes

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \text{.......} + \frac{1}{n(n+1)}$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \text{....} + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \left(1-\frac{1}{n+1}\right)$$

$$= \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \text{ as } n \to \infty$$

$\therefore$ The result holds for x = –1 also.

7. **Prove that:** **(April 2012, April 2010, September 2009)**

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ...;$$

where $-1 \le x \le 1$ **and deduce** $\frac{\pi}{2} = 1 + \frac{1}{2}\frac{1}{3} + \frac{1.3}{2.4}\frac{1}{5} + \frac{1.3.5}{2.4.6}\frac{1}{7}$ ....;

Sol. We have

$$\left(1-x^2\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\left(x^2\right) + \frac{1.3}{2.4}\left(x^2\right)^2 + \frac{1.3.5}{2.4.6}\left(x^2\right)^3 + ...$$

$$...... + \frac{1.3.5...........(2n-1)}{2.4.6............(2n)}.x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1.3.5...........(2n-1)}{2.4.6............(2n)}.x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1.3.5...........(2n-1)}{2.4.6............(2n)}t^n \qquad (\text{where } x^2 = t) \quad (1)$$

Here

$$a_n = \frac{1.3.5.......(2n-1)}{2.4.6....(2n)}$$

$$a_{n+1} = \frac{1.3.5...(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)}$$

$$R = \underset{n\to\infty}{\text{Lt}} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{2n+2}{2n+1} = 1$$

Therefore the series on the R.H.S. is a power series whose radius of convergence is unity.

The series on the R.H.S. is convergent for $|t| < 1$

i.e. for $\left|x^2\right| < 1$,

i.e. for $|x| < 1$

For $x^2 = 1$, the series on the R.H.S. becomes

$$1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + ... + \frac{1.3.5......(2n-1)}{2.4.6........(2n)} + ...$$

Here $u_n = \frac{1.3.5......(2n-1)}{2.4.6........(2n)}$

$$u_{n+1} = \frac{1.3.5......(2n-1)(2n+1)}{2.4.6........(2n)(2n+2)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{2n+2}{2n+1} - 1\right\}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{n(2n+2-2n-1)}{2n+1}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{n}{2n+1} = \frac{1}{2} < 1$$

$\therefore$ By Raabe's test the power series on the R.H.S. is divergent for $x^2 = 1$ i.e. for x = ± 1

Therefore series on the R.H.S. is absolutely convergent for x ∈ (–1, 1) and uniformly convergent in (–λ, λ) where $|\lambda| < 1$. The integrated series will have the same characteristics.

Thus integrating (1) on both sides we get

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1.3.5......(2n-1)}{2.4.6.........(2n)}.\frac{x^{2n+1}}{2n+1} + c$$

For x = 0 we get c = 0

$$\therefore \sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + ...$$

$$+ \frac{1.3.5.....(2n-1)}{2.4.6....(2n)}.\frac{x^{2n+1}}{2n+1} \text{ for } -1 < x < 1 \quad (2)$$

Clearly radius of convergence on the R.H.S. is unity and as such the interval of convergence is –1 < x < 1

By Raabe's test for x = 1

Here $\frac{v_n}{v_{n+1}} = \frac{2n+2}{2n+1}.\frac{2n+3}{2n+1}$

$$\therefore \underset{n\to\infty}{\text{Lt}}\; n\left[\frac{v_n}{v_{n+1}}-1\right] = \underset{n\to\infty}{\text{Lt}}\; n\left[\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}-1\right]$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{n(6n+5)}{4n^2+4n+1}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} > 1$$

$\therefore$ The series on the R.H.S. is convergent for x = 1 by Raabe's test

Similarly the power series on the R.H.S. is convergent for x = –1

Therefore the series is uniformly convergent for x ∈ [–1, 1]

$$\therefore \sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1.3.5......(2n-1)}{2.4.6...........(2n)} \cdot \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1 \quad (2)$$

Put x = 1 on both sides of (2)

$$\sin^{-1}(1) = \frac{\pi}{2} = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7} + ...$$

8. Find the radius of convergence of the series **(April 2012)**

$$x + \frac{1!}{2^2}x^2 + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + ...,$$

Sol. The given series is $\sum_{n=1}^{\infty} \frac{\lfloor n-1}{n^n} x^n$

Here $a_n = \frac{\lfloor n-1}{n^n}$

$$\therefore a_{n+1} = \frac{\lfloor n}{(n+1)^{n+1}}$$

$\therefore$ By Cauchy second theorem on limits

Radius of convergences $= R = \underset{n\to\infty}{Lt} \left|\frac{a_n}{a_{n+1}}\right|$

$$= \underset{n\to\infty}{Lt} \frac{\lfloor n-1}{n^n} \times \frac{(n+1)^{n+1}}{\lfloor n}$$

$$= \underset{n\to\infty}{Lt} \frac{(n+1)^{n+1}}{n \cdot n^n}$$

$$= \underset{n\to\infty}{Lt} \frac{(n+1)^n (n+1)}{n \cdot n^n}$$

$$= \underset{n\to\infty}{Lt} \frac{n^n\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}{n^n}$$

$$= \underset{n\to\infty}{Lt} \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = e$$

9. Show $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ on (–1, 1) and deduce

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ **(September 2011)**

Sol. Take the expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ... \quad (1)$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

Here $a_n = (-1)^n$

$$a_{n+1} = (-1)^{n+1}$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \left|\frac{a_n}{a_{n+1}}\right| = 1$$

$\therefore$ Radius of convergence = 1

and interval of convergence is (–1, 1).

The series on the R.H.S. is a power series which is absolutely convergent for –1 < x < 1. Hence it will be uniformly convergent in (–λ, λ) for $|\lambda| < 1$. The power service does not converge for x = ± 1.

The integrated series will have the same characteristics. Thus integrating both sides w.r.t. x we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + c$$

For x = 0, c = 0

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

For x = 1, the power series on the R.H.S. reduces to

$$1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots\ldots=\Sigma(-1)^{n-1}\left(\frac{1}{n}\right)$$

Which is an alternating series and hence convergent by Leibnitz's test. The above series does converge for x = –1, thus

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots \text{ for } -1<x\le 1 \qquad (2)$$

Deduction

Put x = 1 in (2), we get

$$\log 2=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.$$

10. Find radius of convergence of power series $\sum(5+12i)z^n$.

(September 2011)

Sol. Given series is $\Sigma(5+12i)z^n$

Here $a_n=5+12i$

$$|a_n|=|5+12i|=\sqrt{(5)^2+(12)^2}=\sqrt{169}=13$$

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\lim_{n\to\infty}(13)^{\frac{1}{n}}=13^0=1$$

$$\therefore \text{Radius of convergence}=\frac{1}{\lim_{n\to\infty}|a_n|^{\frac{1}{n}}}=\frac{1}{1}=1$$

⇒ interval of convergence is (–1, 1)

11. Let $\sum_{n=0}^{\infty}a_nx^n$ be power series with radius of convergence R and $f(x)=\sum_{n=0}^{\infty}a_nx^n$ for $|x|<R$. Prove $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{\lfloor n}(x-a)^n$ for $|x-a|<R-|a|$.

(September 2011)

Sol. Suppose $|x-a|<R-|a|$

$$\therefore |x|=|x-a+a|\le|x-a|+|a|<R$$

$$\therefore f(x)=\sum_{n=0}^{\infty}a_nx^n \text{ converges}$$

Consider $f(x)=\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_n(x-a+a)^n=\sum_{n=0}^{\infty}a_n[(x-a)+a]^n$

$$=\sum_{n=0}^{\infty}a_n\sum_{r=0}^{n}{}^nC_ra^{n-r}(x-a)^r$$

$$=\sum_{r=0}^{n}\left[\sum_{n=r}^{\infty}{}^nC_ra_na^{n-r}\right](x-a)^r \qquad (1)$$

Replacing all the quantities by their moduli and taking all terms with positive sign in (1)

$$\sum_{n=0}^{\infty}|a_n|\sum_{r=0}^{n}{}^nC_r|a|^{n-r}|x-a|^r=\sum_{n=0}^{\infty}|a_n|(|x-a|+|a|)^n$$

Which is power series and converges for $|x-a|+|a|<R$

∴ The change in the order of summation is valid

Since $f(x)=a_0+a_1x+a_2x^2+\ldots+a_rx^r+\ldots+a_nx^n+\ldots$

$$\therefore f^{(1)}(x)=a_1+2a_2x+3a_3x^2+\ldots+ra_rx^{r-1}+\ldots+na_nx^{n-1}+\ldots$$

$$f^{(2)}(x)=2a_2+6a_3x+\ldots+r(r-1)a_rx^{r-2}+\ldots+n(n-1)a_nx^{n-2}+\ldots$$

$$f^{(r)}(x)=\lfloor r\, a_r+\ldots.+n(n-1)(n-2)\ldots(n-r+1)a_nx^{n-r}+\ldots.$$

$$=\lfloor r\left[a_r+{}^{r+1}C_ra_{r+1}x+\ldots.+{}^nC_ra_nx^{n-r}+\ldots..\right]$$

$$=\lfloor r\sum_{n=r}^{\infty}{}^nC_ra_nx^{n-r}$$

$$\therefore f^{(r)}(x)=\lfloor r\sum_{n=r}^{\infty}{}^nC_ra_nx^{n-r}$$

At x = a

$$f^r(a)=\lfloor r\sum_{n=r}^{\infty}{}^nC_ra_na^{n-r} \qquad (2)$$

From (1) and (2)

$$\therefore f(x)=\sum_{r=0}^{\infty}\frac{f^{(r)}(a)}{\lfloor r}(x-a)^r \text{ where } |x-a|<R-|a|$$

$$\text{or } f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{\lfloor n}(x-a)^n \text{ for } |x-a|<R-|a|$$

12. Prove $\int_0^x \tan^{-1} x\,dx = \frac{x^2}{1.2} - \frac{x^4}{3.4} + \frac{x^6}{5.6}$............ for $|x|<1$ and show result holds for x = 1 also. **(April 2011)**

Sol. We know $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...for |x| \le 1$

Integrating both sides w.r.t x from 0 to x

$$\int_0^x \tan^{-1} x\,dx = \left[\frac{x^2}{2} - \frac{x^4}{3.4} + \frac{x^6}{5.6} - \frac{x^8}{7.8} - ....\right]_0^x$$

$$= \frac{x^2}{1.2} - \frac{x^4}{3.4} + \frac{x^6}{5.6} - \frac{x^8}{7.8} + ... \qquad (1)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^{2n}}{2n(2n-1)}$$

Taking $x^2 = t$

$$R.H.S. = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{t^n}{2n(2n-1)}$$

Here $u_n = (-1)^{n-1} \cdot \frac{1}{2n(2n-1)}$

$$u_{n+1} = (-1)^n \cdot \frac{1}{(2n+2)(2n+1)}$$

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{(2n-1)(2n)}{(2n+1)(2n+2)}\right| = 1$$

∴ Radius of convergence is unity

∴ $|t| < 1 \Rightarrow |x^2| < 1$

$\Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow x \in (-1, 1)$

∴ interval of convergence is (−1, 1)

At x = 1, the power series on the R.H.S. becomes an alternating series which is convergent by Leibnitz's test. Thus the result holds good for x = 1 also

13. Find radius of convergence of power series $\sum\left(\frac{n\sqrt{2}+i}{1+2ni}\right)x^n$. **(April 2011)**

Sol. $a_n = \frac{n\sqrt{2}+i}{1+2in}$

$$|a_n| = \left|\frac{n\sqrt{2}+i}{1+2in}\right| = \frac{\sqrt{2n^2+1}}{\sqrt{1+4n^2}}$$

$$|a_{n+1}| = \frac{\sqrt{2(n+1)^2+1}}{\sqrt{1+4(n+1)^2}}$$

$$= \sqrt{\frac{2n^2+4n+3}{4n^2+8n+5}}$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \left|\frac{a_n}{a_{n+1}}\right| = \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{2n^2+1}}{\sqrt{1+4n^2}} \times \frac{\sqrt{4n^2+8n+5}}{\sqrt{2n^2+4n+3}}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{2+\frac{1}{n^2}}}{\sqrt{4+\frac{1}{n^2}}} \times \frac{\sqrt{4+\frac{8}{n}+\frac{5}{n^2}}}{\sqrt{2+\frac{4}{n}+\frac{3}{n^2}}}$$

$= 1$

∴ Radius of convergence = R = 1

⇒ interval of convergence = (−1, 1)

14. Define a power series and its radius of convergence. Find radius of convergence of series $x + \frac{x^2}{2^2} + \frac{\lfloor 2 x^3}{3^3} + \frac{\lfloor 3 x^4}{4^4} + ...\infty$. **(September 2010)**

Sol. Definition:-

The series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$ is called a power series in x where the numbers $a_i's$ are dependent on n but not on x and are called the coefficients of power series.

For every power series, there associates a circle called the circle of convergences such that the series $\sum_{n=0}^{\infty} a_n x^n$ converges if x is in interior of the circle and diverges if x is in the exterior of the circle and radius of this circle is called radius of convergence.

Given series is $\sum_{n=1}^{\infty} \frac{\lfloor n-1}{n^n} x^n$

$\therefore a_n = \frac{\lfloor n-1}{n^n}$

$\therefore a_{n+1} = \frac{\lfloor n}{(n+1)^{n+1}}$

$\therefore$ By Cauchy second theorem on limits

Radius of convergence, R = $\underset{n\to\infty}{\text{Lt}} \left|\frac{a_n}{a_{n+1}}\right|$

$= \underset{n\to\infty}{\text{Lt}} \frac{\lfloor n-1}{n^n} \times \frac{(n+1)^{n+1}}{\lfloor n}$

$= \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^{n+1}}{n.n^n}$

$= \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^n (n+1)}{n.n^n}$

$= \underset{n\to\infty}{\text{Lt}} \frac{n^n \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)}{n^n}$

$= \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^{\infty} \left(1+\frac{1}{n}\right) = e$ Ans.

15. **Show** $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ...$ **for** $-1 < x \le 1$ **hence deduce**

$\frac{1}{2}(\sin^{-1} x)^2 = \frac{x^2}{2} + \frac{2}{3}.\frac{x^4}{4} + \frac{2.4}{3.5}\frac{x^6}{6} + ...$ **(September 2010)**

Sol. **First part**

We have

$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}(x^2) + \frac{1.3}{2.4}(x^2)^2 + \frac{1.3.5}{2.4.6}(x^2)^3 + ...$

$...... + \frac{1.3.5...........(2n-1)}{2.4.6............(2n)}.x^{2n}$

$= 1 + \sum_{n=1}^{\infty} \frac{1.3.5...........(2n-1)}{2.4.6............(2n)}.x^{2n}$

$= 1 + \sum_{n=1}^{\infty} \frac{1.3.5...........(2n-1)}{2.4.6............(2n)} t^n$ (where $x^2 = t$) **(1)**

Here

$a_n = \frac{1.3.5.......(2n-1)}{2.4.6....(2n)}$

$a_{n+1} = \frac{1.3.5...(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)}$

$R = \underset{n\to\infty}{\text{Lt}} \left|\frac{a_n}{a_{n+1}}\right|$

$= \underset{n\to\infty}{\text{Lt}} \frac{2n+2}{2n+1} = 1$

Therefore the series on the R.H.S. is a power series whose radius of convergence is unity.

The series on the R.H.S. is convergent for $|t| < 1$

i.e. for $|x^2| < 1$,

i.e. for $|x| < 1$

For $x^2 = 1$, the series on the R.H.S. becomes

$1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + ... + \frac{1.3.5......(2n-1)}{2.4.6........(2n)} + ...$

Here $u_n = \frac{1.3.5......(2n-1)}{2.4.6........(2n)}$

$u_{n+1} = \frac{1.3.5......(2n-1)(2n+1)}{2.4.6........(2n)(2n+2)}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$

$\therefore \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{2n+2}{2n+1} - 1\right\}$

$= \underset{n\to\infty}{\text{Lt}} \frac{n(2n+2-2n-1)}{2n+1}$

$= \underset{n\to\infty}{\text{Lt}} \frac{n}{2n+1} = \frac{1}{2} < 1$

$\therefore$ By Raabe's test the power series on the R.H.S. is divergent for $x^2 = 1$ i.e. for $x = \pm 1$

Therefore series on the R.H.S. is absolutely convergent for $x \in (-1, 1)$ and uniformly convergent in $(-\lambda, \lambda)$ where $|\lambda| < 1$. The integrated series will have the same characteristics.

Thus integrating (1) on both sides we get

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1.3.5......(2n-1)}{2.4.6.........(2n)} \cdot \frac{x^{2n+1}}{2n+1} + c$$

For x = 0 we get c = 0

$$\therefore \sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + ...$$

$$+ \frac{1.3.5.....(2n-1)}{2.4.6....(2n)} \cdot \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 < x < 1 \quad (2)$$

Clearly radius of convergence on the R.H.S. is unity and as such the interval of convergence is $-1 < x < 1$

By Raabe's test for x = 1

Here $\dfrac{v_n}{v_{n+1}} = \dfrac{2n+2}{2n+1} \cdot \dfrac{2n+3}{2n+1}$

$$\therefore \underset{n\to\infty}{\text{Lt}}\, n\left[\frac{v_n}{v_{n+1}} - 1\right] = \underset{n\to\infty}{\text{Lt}}\, n\left[\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} - 1\right]$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{n(6n+5)}{4n^2 + 4n + 1}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{6 + \frac{5}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}} = \frac{6}{4} > 1$$

$\therefore$ The series on the R.H.S. is convergent for x = 1 by Raabe's test

Similarly the power series on the R.H.S. is convergent for x = −1

Therefore the series is uniformly convergent for $x \in [-1, 1]$

$$\therefore \sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1.3.5......(2n-1)}{2.4.6...........(2n)} \cdot \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1 \quad (2)$$

Put x = 1 on both sides of (2)

$$\sin^{-1}(1) = \frac{\pi}{2} = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\cdot\frac{1}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7} + ...$$

Second Part

$$(1-x^2)^{\frac{-1}{2}} = 1 + \frac{1}{2}(x^2) + \frac{1.3}{2.4}(x^2)^2 + \frac{1.3.5}{2.4.6}(x^2)^3 + ...$$

$$.... + \frac{1.3.5.....(2n-1)}{2.4.6......(2n)}(x^2)^n. \text{ For } -1 < x < 1$$

and $\sin^{-1} x = x + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5} + \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7} + ...$

$$.... + \frac{1.3.5.....(2n-1)}{2.4.6......(2n)} \cdot \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1.$$

Both the series are absolutely convergent in (−1, 1). Therefore their Cauchy product will have the same characteristics.

$$\therefore (\sin^{-1} x)(1-x^2)^{\frac{-1}{2}} = \left[1 + \frac{1}{2}(x^2) + \frac{1.3}{2.4}(x^2)^2 + \frac{1.3.5}{2.4.6}(x^2)^3 + ...\right] \times$$

$$\left[x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + ...\right]$$

$$= x + x^3\left(\frac{1}{2} + \frac{1}{6}\right) + x^5\left(\frac{3}{40} + \frac{1}{12} + \frac{3}{8}\right) + ...$$

$$= x + x^3\left(\frac{4}{6}\right) + x^5\left(\frac{9+10+45}{120}\right) + ...$$

$$= x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + ... \quad \text{for } -1 < x < 1$$

Integrating both sides term by term w.r.t. x we have

$$\frac{1}{2}(\sin^{-1} x)^2 = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + ...c_1 \quad \text{for } -1 < x < 1$$

When x = 0, $c_1 = 0$

$$\therefore \frac{1}{2}(\sin^{-1} x)^2 = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + ... + \frac{2.4.......2n}{3.5........(2n+1)}\cdot\frac{x^{2n}}{2n}$$

For $-1 < x < 1$

Here $b_n = \dfrac{2.4.......2n}{3.5........(2n+1)} \cdot \dfrac{1}{2n}$

$$b_{n+1} = \frac{2.4.......2n(2n+2)}{3.5........(2n+1)(2n+3)} \cdot \frac{1}{2n+2}$$

$$\therefore \frac{b_n}{b_{n+1}} = \frac{2n+3}{2n+2} \cdot \frac{2n+2}{2n} = \frac{2n+3}{2n}$$

$$\Rightarrow n\left\{\frac{b_n}{b_{n+1}}-1\right\}=n.\left\{\frac{2n+3}{2n}-1\right\}=n.\frac{3}{2n}=\frac{3}{2}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{b_n}{b_{n+1}}-1\right\}=\frac{3}{2}>1$$

∴ By Raabe's test the power series on the R.H.S. is convergent for $x^2=1$ i.e. for $x=\pm 1$.

$$\therefore\ \frac{1}{2}(\sin^{-1}x)^2=\frac{x^2}{2}+\frac{2}{3}.\frac{x^4}{4}+\frac{2.4}{3.5}.\frac{x^6}{6}+\ldots\ldots \text{ for } -1\le x\le 1.$$

16. Prove that powers series $\sum a_n x^n$ is uniformly convergent for $|x|\le r<R$ where R is radius of convergence. **(September 2010)**

Sol. Let r' be a number between r and R.

Since the given series is convergent for $|x|=r'$.

∴ There is a number k independent of n, so that $|a_n r'^n|<k\ \forall n$

Hence, we have for (1)

$$|x|\le r\quad |a_n x^n| \quad \left|a_n r'^n \frac{x}{r'}\right|^n \quad k\ \frac{r}{r'}^{\,n} \quad \text{by (1) and (2)} \qquad (2)$$

$$\Rightarrow |a_n x^n|<k\left(\frac{r}{r'}\right)^n=M_n$$

Since, the series $k\sum\left(\frac{r}{r'}\right)^n$ is G.P. series with common ratio $\frac{r}{r'}<1$

$$\Rightarrow \sum M_n=k\sum\left(\frac{r}{r'}\right)^n \text{ is convergent.}$$

∴ By Weierstrass-M test $\sum a_n x^n$ is uniformly convergent for $|x|<r<R$

17. Find the radius of convergence and interval of convergence of power series $\sum\frac{(-1)^n}{n}(z-2i)^n$. **(September 2009)**

Sol. The given series is $\sum\frac{(-1)^n}{n}(z-2i)^n$

$$=\sum\frac{(-1)^n}{n}.t^n \qquad \text{(Taking } z-2i=t)$$

Here $u_n=\frac{(-1)^n}{n}, u_{n+1}=\frac{(-1)^{n+1}}{n+1}$

$$\therefore\ \underset{n\to\infty}{\text{Lt}}\left|\frac{u_n}{u_{n+1}}\right|=\underset{n\to\infty}{\text{Lt}}\frac{n+1}{n}=1$$

⇒ Radius of convergence of the series in t = 1

Now $|t|<1$

$\Rightarrow |z-2i|<1$

$\Rightarrow -1<z-2i<1$

$\Rightarrow -1+2i<z<1+2i$

∴ interval of convergence of the given series is $(-1+2i, 1+2i)$

18 Find the radius of convergence and interval of convergence of the power series. $\sum\left(\frac{n\sqrt{2}+i}{1+2in}\right)x^n$. **(April 2009)**

Sol. Here $a_n=\frac{n\sqrt{2}+i}{1+2in}$

$$|a_n|=\left|\frac{n\sqrt{2}+i}{1+2in}\right|$$

$$=\frac{|n\sqrt{2}+i|}{|1+2in|}=\frac{\sqrt{2n^2+1}}{\sqrt{1+4n^2}}$$

$$|a_{n+1}|=\frac{\sqrt{2(n+1)^2+1}}{\sqrt{1+4(n+1)^2}}$$

$$=\sqrt{\frac{2n^2+4n+3}{4n^2+8n+5}}$$

$$\therefore\ \underset{n\to\infty}{\text{Lt}}\left|\frac{a_n}{a_{n+1}}\right|=\underset{n\to\infty}{\text{Lt}}\frac{\sqrt{2n^2+1}}{\sqrt{1+4n^2}}\times\frac{\sqrt{4n^2+8n+5}}{\sqrt{2n^2+4n+3}}$$

$$=\underset{n\to\infty}{\text{Lt}}\frac{\sqrt{2+\frac{1}{n^2}}}{\sqrt{4+\frac{1}{n^2}}}\times\frac{\sqrt{4+\frac{8}{n}+\frac{5}{n^2}}}{\sqrt{2+\frac{4}{n}+\frac{3}{n^2}}}$$

$=1$

∴ Radius of convergence = R = 1

⇒ interval of convergence = (–1, 1)

19. **Show that:**

(i) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots\ldots, -1 \le x < 1$

(ii) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots$

(iii) $\frac{1}{2}[\log(1-x)]^2 = \frac{x^2}{2} + \left(1+\frac{1}{2}\right)\frac{x^3}{3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4} + \ldots\ldots$ **(April 2009)**

For $-1 \le x < 1$.

Sol. (i)Take the expansion

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots$ (1)

Which being a G.P. series will converge for $|x| < 1$ i.e. for $-1 < x < 1$

The series on the R.H.S. is a power series which is absolutely convergent for $-1 < x < 1$. Hence it will be uniformly convergent in $(-\lambda, \lambda)$ for $|\lambda| < 1$. The power series does not converge for $x = \pm 1$.

The integrated series will have the same characteristics. Thus integrating both sides w.r.t. we have:

$$\frac{\log(1-x)}{-1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots + c$$

$$\Rightarrow \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\ldots - c$$

For x = 0, c = 0

$$\therefore \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots$$

For x = –1, the power series on the R.H.S. reduces to

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots = \sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n}\right)$$

Which is an alternating series and hence convergent by Leibnitz's test. The above series does not converge for x = 1. Thus

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots \text{ for } -1 \le x < 1 \quad (2)$$

(ii) For x = –1

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots$$

From (1) and (2) both the power series on the R.H.S. convergeabsolutely in (–1, 1) therefore their Cauchy product will be also convergent in (–1, 1)

$$\therefore (1-x)^{-1}\log(1-x) = [1 + x + x^2 + x^3 + \ldots\ldots]\left[-x - \frac{x^2}{2} - \frac{x^3}{3} \ldots\ldots\ldots\right]$$

$$= -x - x^2\left(1+\frac{1}{2}\right) - x^3\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots\ldots\ldots\ldots \text{ for } -1 < x < 1$$

Integrating both sides from 0 to x.

$$-\frac{1}{2}[\log(1-x)]^2 = -\frac{x^2}{2} - \frac{x^3}{3}\left(1+\frac{1}{2}\right) - \frac{x^4}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots\ldots \text{ for } -1 < x < 1$$

$$\Rightarrow \frac{1}{2}[\log(1-x)]^2 = \frac{x^2}{2} + \frac{x^3}{3}\left(1+\frac{1}{2}\right) + \frac{x^4}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots\ldots$$

for $-1 < x < 1$

For x = –1, the power series on the R.H.S. reduces to

$$\frac{1}{2} - \frac{1}{3}\left(1+\frac{1}{2}\right) + \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots\ldots\ldots\ldots$$

Which is an alternating series and is convergent by Leibnitz's test

The power series on the R.H.S. does not converge for x = 1

$$\therefore \frac{1}{2}[\log(1-x)]^2 = \frac{x^2}{2} + \frac{x^3}{3}\left(1+\frac{1}{2}\right) + \frac{x^4}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots\ldots$$

for $-1 \le x < 1$

9

FOURIER SERIES

1. Show that Fourier series which converges to $f(x)$ in $[-\pi,\pi]$

where $f(x)=\begin{cases} x+x^2 & \text{if } -\pi<x<\pi \\ \pi^2 & \text{if } x=\pm\pi \end{cases}$ **is** $\frac{\pi^2}{3}+4(-1)^n\left[\frac{\cos nx}{n^2}-\frac{\sin nx}{2n}\right]$.

Hence deduce that $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+........$ **(September 2013)**

Sol. $f(x)=x+x^2$ and $x\in[-\pi,\pi]$

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

By Euler's formulae

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}xdx+\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx$$

$$=0+\frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[x\cos nx+x^2\cos nx\right]dx$$

$$=\frac{1}{\pi}\left[0+2\int_0^{\pi}x^2\cos nxdx\right]$$ $\because x\cos nx$ is an odd and $x^2\cos nx$ is an even function

$$=\frac{2}{\pi}\left[x^2.\frac{\sin nx}{n}\Big|_0^{\pi}-\int_0^{\pi}2x\frac{\sin nx}{n}dx\right]$$

$$=\frac{-4}{\pi n}\left[x\left\{-\frac{\cos nx}{n}\right\}_0^{\pi}-\int_0^{\pi}1.\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$=\frac{4}{\pi n^2}[\pi\cos n\pi+0]=\frac{4}{n^2}(-1)^n$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[x\sin nx+x^2\sin nx\right]dx$$

$$=\frac{1}{\pi}\left[2\int_0^{\pi}x\sin nxdx+0\right]$$ $\because x\sin nx$ is an even and $x^2\sin nx$ is an odd function

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)_0^{\pi}-\int_0^{\pi}1.\left(\frac{\cos nx}{-n}\right)dx\right]$$

$$=-\frac{2\pi\cos n\pi}{n\pi}=-\frac{2}{n}(-1)^n$$

$$\therefore f(x)=x+x^2=\frac{1}{2}.\frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx+\sum_{n=1}^{\infty}\left\{-\frac{2}{n}(-1)^n\right\}\sin nx$$

$$=\frac{\pi^2}{3}+4\left\{-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+....\right\}+2\left\{\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+........\right\}$$

$$=\frac{\pi^2}{3}-4\left\{\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+....\right\}+2\left\{\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+........\right\}$$

$$f(x)=\frac{\pi^2}{3}+4\sum(-1)^n\left[\frac{\cos nx}{n^2}-\frac{\sin nx}{2n}\right]$$

Put $x=\pm n$

$$f(\pm\pi)=\frac{\pi^2}{3}+4\sum(-1)^n\left[\frac{\cos nx}{n^2}-0\right]$$

$$\Rightarrow \pi^2=\frac{\pi^2}{3}+4\sum\frac{1}{n^2} \qquad \{\because f(x)=\pi^2 \text{ for } x^2=\pm\pi(given)\}$$

$$\Rightarrow \frac{2\pi^2}{3}=4\sum\frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....$$

2. Express "sin x" as cosine series when $0<x<\pi$. **(September 2013)**

Sol. Let $f(x)=\sin x$, $x\in(0,\pi)$

Clearly $f(x)$ is bounded, integrable and monotonic in $(0,\ \pi)$

Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

Now $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}(\sin x)dx=\frac{2}{\pi}(-\cos x)_0^{\pi}=\frac{4}{\pi}$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx dx=\frac{2}{\pi}\int_0^{\pi}\sin x.\cos nx dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]dx$$

$$=\frac{1}{\pi}\left[-\frac{\cos(n+1)}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right],\ n\neq 1$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n}{n+1}-\frac{(-1)^n}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{1}{\pi}\left[\{(-1)^n+1\}\left\{\frac{1}{n+1}-\frac{1}{n-1}\right\}\right]$$

$$=\frac{-2}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right],\ n\neq 1$$

In particular for $n=1$

$$a_1=\frac{2}{\pi}\int_0^{\pi}\sin x\cos x dx=\frac{1}{2}\int_0^{\pi}\sin 2x dx=0$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi}\sin x\sin nx dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\cos x(n-1)-\cos x(n+1)dx]$$

$$=\frac{1}{\pi}\left[\frac{\sin x(n-1)}{n-1}-\frac{\sin x(n+1)}{n+1}\right]_0^{\pi}=0,\ n\neq 1$$

In particular for $n=1$

$$b_1=\frac{2}{\pi}\int_0^{\pi}\sin x.\sin x=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin^2 x$$

$$=\frac{2}{\pi}.\frac{1}{2}.\frac{\pi}{2}=\frac{1}{2}$$

$$f(x)=\frac{1}{2}a_0+a_1\cos x+b_1\sin x+\sum_{n=2}^{\infty}a_n\cos nx+\sum_{n=2}^{\infty}a_n\sin nx$$

$$\Rightarrow \sin x=\frac{1}{2}\left(\frac{4}{\pi}\right)+0+b_1\sin x-\frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx$$

$$=\frac{2}{\pi}+\frac{1}{2}\sin x-\frac{4}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\ldots\ldots\right]$$

$$\Rightarrow \frac{\sin x}{2}=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\ldots\ldots\right]$$

Hence $\sin x=\frac{4}{\pi}-\frac{8}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\ldots\ldots\right]$

3. **Find the Fourier series expansion of $f(x)=x\sin x,\ x\in(-\pi,\pi)$. Hence deduce that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots\ldots=\frac{\pi-2}{4}$.** **(April 2013, September 2012)**

Sol. Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

Because $f(x)=x\sin x$ is an even function $\therefore$ $b_n=0$

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x dx=\frac{2}{\pi}\int_0^{\pi}x\sin x dx$

$$=\frac{2}{\pi}\left[x(-\cos x)|_0^{\pi}+\int_0^{\pi}\cos x dx\right]$$

$$=\frac{2}{\pi}\left[(-\pi\cos\pi+0)+[\sin x]_0^{\pi}\right]$$

$$=\frac{2}{\pi}[-\pi(-1)+0+(0-0)]$$

$$=\frac{2}{\pi}[\pi]=2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos nx dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nx dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[x\sin(n+1)x-x\sin(n-1)x]dx$$

$$=\frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{n+1}\Bigg|_0^{\pi}+\int_0^{\pi}\frac{\cos(n+1)x}{n+1}dx+\frac{x\cos(n-1)x}{n-1}\Bigg|_0^{\pi}-\int_0^{\pi}\frac{\cos(n-1)x}{n-1}dx\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{n+1}+\frac{\pi\cos(n-1)\pi}{n-1}\right], n\neq 1$$

$$=\frac{\pi}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right]=(-1)^n\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=(-1)^n\left[\frac{-2}{n^2-1}\right]=-\frac{2(-1)^n}{n^2-1}, n\neq 1$$

In particular when n = 1

$$a_1=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos x\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos x\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}x\sin 2x\,dx$$

$$=\frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)\Bigg|_0^{\pi}-\int_0^{\pi}\left(-\frac{\cos 2x}{2}\right)dx\right]$$

$$=\frac{-1}{\pi}\left[\frac{\pi}{2}.\cos 2\pi\right]=-\frac{1}{2}$$

Thus $f(x)=\frac{1}{2}a_0+a_1\cos x+\sum_{n=2}^{\infty}a_n\cos nx$

$$\therefore\ x\sin x=\frac{1}{2}(2)-\frac{\cos x}{2}-2.\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

$$=1-\frac{\cos x}{2}-2\left[\frac{\cos 2x}{3}-\frac{\cos 3x}{8}+\frac{\cos 4x}{15}+\ldots\right]$$

Deduction

Put $x=\frac{\pi}{2}$

$$\frac{\pi}{2}=1-2\left[\frac{-1}{3}+\frac{1}{15}-\frac{1}{35}+\ldots.\right]$$

$$=1+\frac{2}{3}-\frac{2}{3.5}+\frac{2}{5.7}\ldots\ldots$$

Dividing both sides by 2

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}+\ldots..$$

OR

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}+\ldots=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

4. Find the Fourier series of $f(x)=x^2$ in $[-\pi,\pi]$ and deduce the following:

(i) $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

(ii) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

(iii) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$ **(April 2013, April 2010, April 2009)**

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

By Euler's Formulae

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2dx \qquad \{\because x^2 \text{ is an even function}\}$$

$$=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2\pi^3}{3\pi}=\frac{2}{3}\pi^2$$

Now $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx$

$$=\frac{2}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx \qquad \{\because x^2\cos nx \text{ is an even function}\}$$

$$=\frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\pi^2\left(\frac{\sin n\pi}{n}\right)+\frac{2\pi\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}-(0+0-0)\right]$$

$$=\frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]=\frac{4(-1)^n}{n^2} \qquad \{\because \sin n\pi=0 \text{ and } \cos n\pi=(-1)^n\}$$

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\,dx$

$$= \frac{1}{\pi}(0) = 0 \qquad \{\because x^2 \sin nx \text{ is an odd function}\}$$

Using above results, we get

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\dots\dots\right]$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \dots\dots\right] \qquad (1)$$

(i) Putting x = π in (1)

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}\dots\dots\right]$$

$$\therefore \pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\dots\dots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots \quad \textbf{Ans.} \qquad (2)$$

(ii) Putting x = 0 in (1) we get

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots \quad \textbf{Ans.} \qquad (3)$$

(iii) Adding (2) and (3)

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\dots\right] = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\dots = \frac{\pi^2}{8} \quad \textbf{Ans.}$$

5. Obtain the Fourier series for $f(x) = x^3 \ (-\pi < x < \pi)$. **(September 2012)**

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 dx = 0 \qquad (x^3 \text{ is odd function})$$

$$\text{Now } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos nx dx = 0 \qquad (x^3 \cos nx \text{ is odd function})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx dx \qquad (x^3 \sin nx \text{ is even function})$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - \int 3x^2\left(\frac{-\cos nx}{n}\right)dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{3}{n}\int x^2 \cos nx dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{3}{n}\left[\frac{x^2 \sin nx}{n} - \int \frac{2x \sin nx}{n} dx\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{3x^2}{n^2}\sin nx - \frac{6}{n^2}\int x \sin nx dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{3x^2}{n^2}\sin nx - \frac{6}{n^2}\left[x\left(\frac{-\cos nx}{n}\right) - \int \frac{(-\cos nx)}{n} dx\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{3x^2}{n^2}\sin nx + \frac{6x}{n^3}\cos nx - \frac{6}{n^3}\int \cos nx dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^3 \cos nx}{n} + \frac{3x^2}{n^2}\sin nx + \frac{6x}{n^3}\cos nx - \frac{6}{n^4}\sin nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^3(-1)^n}{n} + \frac{6\pi(-1)^n}{n^3}\right]$$

$$= \frac{2\pi(-1)^n}{\pi}\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right]$$

$$= \frac{2(-1)^n(6 - n^2\pi^2)}{n^3}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\left[\frac{2(-1)^n(6 - n^2\pi^2)}{n^3}\right]\sin nx$$

Which is the required Fourier series

6. Obtain the Fourier Series for $f(x)=\begin{cases}\pi x & ; \ 0\le x\le 1\\ \pi(2-x) & ; \ 1\le x\le 2\end{cases}$ **(September 2012)**

Sol. Here, $l=1$

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x+\sum_{n=1}^{\infty}b_n\sin n\pi x$

$$a_0=\int_0^2 f(x)\,dx=\int_0^1\pi x\,dx+\int_1^2\pi(2-x)\,dx$$

$$=\pi\left[\frac{x^2}{2}\right]_0^1+\pi\left[2x-\frac{x^2}{2}\right]_1^2$$

$$=\pi\left[\frac{1}{2}\right]+\pi\left[(4-2)-\left(2-\frac{1}{2}\right)\right]=\pi$$

$$a_n=\int_0^2 f(x)\cos n\pi x\,dx$$

$$=\int_0^1\pi x\cos n\pi x\,dx+\int_1^2\pi(2-x)\cos n\pi x\,dx$$

In the second integral

Take $2-x=t\Rightarrow dx=-dt$

$x=2\Rightarrow t=0$

$x=1\Rightarrow t=1$

$$\therefore \int_1^2\pi(2-x)\cos n\pi x\,dx$$

$$=\int_1^0\pi(t)\cos(2n\pi-n\pi t)(-dt)$$

$$=\pi\int_0^1 t\cos n\pi t\,dt=\pi\int_0^1 x\cos n\pi x\,dx$$

$$=\int_0^1\pi x\cos n\pi x\,dx+\int_0^1\pi x\cos n\pi x\,dx$$

$$=2\int_0^1\pi x\cos n\pi dx$$

$$=2\left[\pi x.\frac{\sin n\pi x}{n\pi}-\pi\left(-\frac{\cos n\pi x}{\pi^2n^2}\right)\right]_0^1$$

$$=2\left(\frac{\cos n\pi}{n^2\pi}-\frac{1}{n^2\pi}\right)$$

$$=\frac{2}{n^2\pi}(\cos n\pi-1)=\frac{2}{n^2\pi}\{(-1)^n-1\}$$

$$=\begin{cases}\frac{-4}{n^2\pi}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

(By the above substitution)

$$And \quad b_n=\int_0^2 f(x)\sin n\pi x\,dx$$

$$\therefore \int_1^2\pi(2-x)\sin n\pi x\,dx$$

$$=\int_1^0\pi t\sin(2n\pi-n\pi t)(-dt)$$

$$=-\int_0^1\pi t\sin n\pi t\,dt$$

$$=-\int_0^1\pi x\sin n\pi x\,dx$$

$$=\int_0^1\pi x\sin n\pi x\,dx+\int_1^2\pi(2-x)\sin n\pi x\,dx$$

$$=\int_0^1\pi x\sin n\pi x\,dx-\int_0^1\pi x\sin n\pi x\,dx=0$$

$$=0$$

$$\therefore f(x)=\frac{1}{2}(\pi)+\sum_{n=0}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n-1\right]\cos n\pi x$$

$$=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos\pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+.....\right]$$

7. Show that the Fourier series expansion of $f(x)=x-x^2, x\in(-\pi,\pi)$ **is** $x-x^2=\frac{-\pi^2}{3}+\left[\sum_{n=1}^{\infty}\left\{(-1)^{n+1}\frac{4}{n^2}\cos nx+(-1)^{n+1}\frac{2}{n}\sin nx\right\}\right].$ **(April 2012)**

Sol. Let $f(x)=x-x^2=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ (By def.)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,dx-\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx$$

$$=0+\frac{1}{\pi}\left[-\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\left(-\frac{\pi^3}{3}\right)-\left(+\frac{\pi^3}{3}\right)\right]$$ $\because$ x is an odd and x^2 is an even function

$$=-\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$\because$ x cos nx is an odd and x^2cos nx is an even function

$$= 0 - \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= -\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi} 2x\left(\frac{\sin nx}{n}\right)dx\right]$$

$$= \frac{4}{n\pi}\int_0^{\pi} x.\sin nx\,dx$$

$$= \frac{4}{n\pi}\left[x\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right] = -\frac{4}{n^2\pi}\left[\pi\cos n\pi - 0 - \int_0^{\pi}\cos nx\,dx\right]$$

$$= -\frac{4}{n^2\pi}\left[\pi\cos n\pi - \left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{-4}{n^2\pi}\left[\pi\cos n\pi\right] = \frac{-4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx + 0$$

$\because$ x sin nx is an even function and x^2 sin nx is an odd function

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}dx\right]$$

$$= \frac{2}{n\pi}\left[-\pi\cos n\pi + 0 + \left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{2}{n\pi}\left[-\pi\cos n\pi + 0\right] = -\frac{2}{n}\cos n\pi = -2\frac{(-1)^n}{n}$$

$$\therefore\ f(x) = x - x^2 = -\frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

8. Show that the Fourier series expansion of $f(x) = \sin mx, x \in (-\pi, \pi)$ and $m \in R - Z$ is $\frac{2}{\pi}\sin m\pi\left[\sum_{r=1}^{\infty}(-1)^{r+1}\frac{r\sin rx}{r^2 - m^2}\right]$. **(April 2012)**

Sol. According to Euler's formulae

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\cos nx + \sum_{n=1}^{\infty}b_n\sin nx$$

Since $f(x) = \sin mx$ is an odd function of x.

Therefore $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin mx\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin mx\sin nx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\cos(n-m)x - \cos(n+m)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-m)\pi}{n-m} - \frac{\sin(n+m)\pi}{n+m}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi\cos m\pi - \cos n\pi\sin m\pi}{n-m} - \frac{\sin n\pi\cos m\pi + \cos n\pi\sin m\pi}{n+m}\right]$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^n\sin m\pi}{n-m} + \frac{(-1)^n\sin m\pi}{n+m}\right]$$

$\because \cos n\pi = (-1)^n$, $\sin n\pi = 0$

$$= -\frac{(-1)^n\sin m\pi}{\pi}\left[\frac{1}{n-m} + \frac{1}{n+m}\right]$$

$$= (-1)^{n+1}\frac{2n\sin m\pi}{\pi\left(n^2 - m^2\right)}$$

$$\therefore\ f(x) = \sin mx = \frac{2\sin m\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2 - m^2}n\sin nx$$

$$= \frac{2\sin m\pi}{\pi}\left(\frac{\sin x}{1^2 - m^2} - \frac{2\sin 2x}{2^2 - m^2} + \frac{3\sin 3x}{3^2 - m^2} + \dots\dots\right)$$

or $f(x) = \sin mx = \frac{2}{\pi}\sin m\pi \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2 - m^2} r.\sin rx$

9 Define a Fourier series and the method to find Fourier coefficients. **(September 2011)**

Sol. Euler's Formula:

According to Euler Formula, Fourier series for function f(x) in the interval c< x <c + 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \textbf{(1)}$$

In finding the coefficients a_0, a_n, b_n we assume that series or RHS can be integrated term by term in given interval c< x <c + 2π

To find a_0

Integrate both sides of (1) w.r.t. x between limits c to c + 2π

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(x)_c^{c+2\pi} + 0 + 0$$

$$\left(\int_c^{c+2\pi} \sin nx\,dx = 0 \quad n \neq 0 \quad \text{and} \quad \int_c^{c+2\pi} \cos nx\,dx = 0 \quad n \neq 0\right)$$

$$= \frac{a_0}{2}(c + 2\pi - c) = \frac{a_0}{21}(2\pi) = \pi a_0$$

$$\therefore a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

To find a_n

Multiply both sides of (1) by cos nx and integrate w.r.t. x between limits c to c + 2π

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= \frac{a_0}{2}(0) + \int_c^{c+2\pi} a_n \cos^2 nx\,dx + \int_c^{c+2\pi} b_n \sin nx \cos nx\,dx$$

$$\left(\int_c^{c+2\pi} \cos nx\,dx = 0 \quad \int_c^{c+2\pi} \cos mx \cos nx\,dx = 0 \quad m \neq n \quad \& \right.$$

$$\left.\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0 \quad m \neq n\right)$$

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = a_n \int_c^{c+2\pi}\left(\frac{1+\cos 2nx}{2}\right)dx + 0 \quad \left(\int_c^{c+2\pi} \sin 2nx\,dx = 0\right)$$

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_n}{2}\left[x + \frac{\sin 2nx}{2n}\right]_c^{c+2\pi}$$

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_n}{2}(2\pi)$$

$$\therefore a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

To find b_n

Multiply both sides of (1) by sin nx and integrate w.r.t. x between limits c to c + 2π

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= \frac{a_0}{2}(0) + \int_c^{c+2\pi} a_n \sin nx \cos nx\,dx + \int_c^{c+2\pi} b_n \sin^2 nx\,dx$$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{b_n}{2}\left[x - \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \frac{b_n}{2}[c + 2\pi - c]$$

$$\therefore \int_c^{c+2\pi} f(x)\sin dx = 0 + 0 + b_n \int_c^{c+2\pi} \frac{(1 - cos 2nx)\,dx}{2}$$

$$\left(\int_c^{c+2\pi} \sin 2nx\,dx = 0\right) \int_c^{c+2\pi} f(x)\sin dx = \frac{b_n}{2}\left[x - \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \frac{b_n}{2}[c + 2\pi - c]$$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{b_n}{2}(2\pi)$$

$a_1 \cos x + b_1 \sin x = ($

$$\therefore \frac{b_n}{2} = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Hence, we conclude that By Euler's Formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

10. Find Fourier series expansion of $f(x) = |x|$ on $[-\pi, \pi]$.

(September 2011)

Sol. $f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ is an even function hence $b_n = 0$

By Euler's Formulae

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\dfrac{4}{\pi n^2} & \text{if n is odd} \end{cases}$$

$$\therefore f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$$

11. Obtain Fourier series to represent $f(x) = \frac{1}{4}(\pi - x)^2$ on $[0, 2\pi]$.

(September 2011, September 2010)

Sol. Let $f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ (1)

We have by Euler's formulae.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{-2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{1}{4\pi}\left[\frac{2\pi^3}{3}\right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - \left(2(\pi - x)(-1)\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left((-\pi)^2 . \frac{\sin 2n\pi}{n} - \frac{2(-\pi)\cos 2n\pi}{n^2} - \frac{2\sin 2n\pi}{n^3}\right) - \left(0 - 2\pi.\frac{1}{n^2} - 0\right)\right]$$

$$= \frac{1}{4\pi}\left[0 + \frac{2\pi}{n^2} - 0 + \frac{2\pi}{n^2}\right]$$

$$=\frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right]=\frac{1}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right)-(2)(\pi-x)(-1)\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(\frac{-\pi^2\cos 2n\pi}{n}-0+\frac{2\cos 2n\pi}{n^3}\right)-\left(\frac{-\pi^2\cos 0}{n}-0+\frac{2\cos 0}{n^3}\right)\right]$$

$$=\frac{1}{4\pi}\left(\frac{-\pi^2}{n}+\frac{2}{n^3}+\frac{\pi^2}{n}-\frac{2}{n^3}\right)=0$$

Substituting the values in (1)

$$f(x)=\frac{1}{4}(\pi-x)^2=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{1}{4}(\pi-x)^2=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\ldots$$

12. Find Fourier series of $f(x)=x$ on $[-\pi,\pi]$. **(April 2011)**

Sol. Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

Because $f(x)=x$ is an odd function therefore xcosnx will also be an odd function. As such a_0 and a_n will be both vanish and the Fourier series will be only a sine series

$$\therefore\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x.\sin nx=\frac{2}{\pi}\int_0^{\pi}x\sin nx dx$$

$$=\frac{2}{\pi}\left[\left(x.\frac{-\cos nx}{n}\right)_0^{\pi}+\int_0^{\pi}\frac{\cos nx}{n}dx\right]$$

$$=\frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+0\right)+\left(\frac{\sin n\pi}{n^2}\right)_0^{\pi}\right]$$

$$=-\frac{2}{n\pi}(\pi\cos n\pi)=-\frac{2}{n}(-1)^n$$

$$\therefore\ f(x)=x=0+0-\sum_{n=1}^{\infty}\frac{2}{n}(-1)^n\sin nx$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

$$\Rightarrow x=2\left[\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 4x}{4}+\ldots\ldots\right]$$

13. $f(x)=x+x^2$ on $(-\pi,\pi)$, find Fourier Series expansion and deduce.

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\infty.$$

(April 2011, April 2010)

Sol. $f(x)=x+x^2$ and $x\in[-\pi,\pi]$

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

By Euler's formulae

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x dx+\frac{1}{\pi}\int_{-\pi}^{\pi}x^2 dx$$

$$=0+\frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[x\cos nx+x^2\cos nx\right]dx$$

$$=\frac{1}{\pi}\left[0+2.\int_0^{\pi}x^2\cos nx dx\right]$$ $\because$ $x\cos nx$ is an odd and $x^2\cos nx$ is an even function

$$=\frac{2}{\pi}\left[x^2.\frac{\sin nx}{n}\Big|_0^{\pi}-\int_0^{\pi}2x\frac{\sin nx}{n}dx\right]$$

$$=\frac{-4}{\pi n}\left[\left\{x\left(-\frac{\cos nx}{n}\right)\right\}_0^{\pi}-\int_0^{\pi}1.\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$=-\frac{4}{\pi n}\left[-\frac{\pi\cos n\pi}{n}+0+\frac{1}{n^2}(\sin n\pi)_0^{\pi}\right]$$

$$=\frac{4}{\pi n^2}[\pi\cos nx+0]=\frac{4}{n^2}(-1)^n$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[x\sin nx + x^2 \sin nx\right]dx$$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx dx + 0\right] \quad \left| \because \; x\sin nx \text{ is an even and } x^2\sin nx \text{ is an odd function}\right.$$

bn = 0, an = 0 = a0

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1.\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n} + 0 + \frac{1}{n^2}\left[\sin n\pi\right]_0^{\pi}\right]$$

$$= -\frac{2\pi\cos n\pi}{n\pi} = -\frac{2}{n}(-1)^n$$

$$\therefore f(x) = x + x^2 = \frac{1}{2}.\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + \sum_{n=1}^{\infty}\left\{-\frac{2}{n}(-1)^n\right\}\sin nx$$

$$= \frac{\pi^2}{3} + 4\left\{-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\right\}$$

$$+2\left\{\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right\}$$

$$\Rightarrow x + x^2 = \frac{\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right\}$$

$$+2\left\{\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\ldots\ldots\right\} \qquad (1)$$

Deduction

Put x = 0 in (1)

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots\right)$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots\ldots$$

14. Find Fourier expansion of $f(x) = x - x^3$ on (–1, 1). **(April 2011)**

Sol. Here $l = 1$

Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Since the given function is an odd function therefore a_0 and a_n will both vanish

Now $b_n = \frac{1}{l}\int_{-1}^{1} f(x)\sin n\pi x dx$

$$= \frac{1}{1}\int_{-1}^{1}(x - x^3)\sin n\pi x dx$$

$$= 2\int_0^1 (x - x^3)\sin n\pi x dx$$

$$= 2\left[(x - x^3)\left(-\frac{\cos n\pi x}{n\pi}\right)\Big|_0^1 - \int_0^1 (1 - 3x^2)\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right]$$

$$= \frac{2}{n\pi}\int_0^1 (1 - 3x^2)\cos n\pi x dx$$

$$= \frac{2}{n\pi}\left[(1 - 3x^2)\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^1 (-6x)\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= \frac{12}{n^2\pi^2}\int_0^1 x.\sin n\pi x dx$$

$$= \frac{12}{n^2\pi^2}\left[\left\{x\left(-\frac{\cos n\pi x}{n\pi}\right)\right\}_0^1 - \int_0^1\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right]$$

$$= \frac{12}{n^2\pi^2}\left[-\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\Big|_0^1\right]$$

$$= -\frac{(12)(-1)^n}{n^3\pi^3}$$

$$\therefore f(x) = -\frac{(12)}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x$$

$$= \frac{(12)}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin n\pi x$$

$$= \frac{12}{\pi^3}\left[\frac{\sin \pi x}{1^3} - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} + \ldots\ldots\ldots\right]$$

15. Expand in series of sines and cosines of multiple angle of x, the periodic function f with period 2π defined as $f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \le x < \pi \end{cases}$

Also calculate the sum of series at $x = \frac{\pi}{2}, \pm\pi$. **(September 2010, 2009)**

Sol. $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)dx+\int_{0}^{\pi}(1)dx\right]$$

$$=\frac{1}{\pi}\left[[-x]_{-\pi}^{0}+[-x]_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[-[0+\pi]+[\pi-0]\right]=\frac{1}{\pi}(-\pi+\pi)=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\cos nxdx+\int_{0}^{\pi}\cos nxdx\right]$$

$$=\frac{1}{\pi}\left[-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}[0+0]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\sin nxdx+\int_{0}^{\pi}\sin nxdx\right]$$

$$=\frac{1}{\pi}\left[-\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0}+\left(\frac{-\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0}-\left(\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{1}{n}-\frac{\cos n\pi}{n}\right)-\left(\frac{\cos n\pi}{n}-\frac{1}{n}\right)\right]$$

$$=\frac{1}{\pi}\left[\frac{2}{n}-\frac{2\cos n\pi}{n}\right]$$

$$=\frac{2}{\pi n}\left(1-(-1)^n\right)=\begin{cases}0 & \text{if n is even}\\ \frac{4}{\pi n} & \text{if n is odd}\end{cases}$$

$$\therefore\ f(x)=\sum_{n=1}^{\infty}\frac{2}{\pi n}\left(1-(-1)^n\right)\sin nx$$

$$=\frac{2}{\pi}\left[\frac{2}{1}\sin x+\frac{2}{3}\sin 3x+\frac{2}{5}\sin 5x+----\right]$$

$$=\frac{4}{\pi}\left[\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+----\right]$$

Put $x=\frac{\pi}{2}$ in the above series

$$1=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-----\right]$$

$$\Rightarrow\ \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-----$$

At $x=\pm\pi$, $f(x)$ is discontinuous

$\therefore$ sum of series at $x=\pm\pi$ is $\frac{1}{2}\left[f(\pi-)+f(-\pi+)\right]$

$$\frac{1}{2}\left[1+(-1)\right]=0$$

16. Find a series of sine and cosines of multiples of x which represent $\frac{\pi}{2\sinh\pi}e^x$ in $(-\pi,\pi)$. **(September 2009)**

Sol. Consider $f(x)=e^x$ in $(-\pi,\pi)$

$$\therefore\ f(x)=e^x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

then $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x dx=\frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[e^{\pi}-e^{-\pi}\right]$

$$\Rightarrow\ a_0=\frac{2\sinh\pi}{\pi}$$

$e^{\pi}-e^{-\pi}=\frac{2\sinh\pi}{\pi}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\cos nxdx=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi(n^2+1)}\left[e^{\pi}\cos n\pi-e^{-\pi}\cos n\pi\right]$$

$$=\frac{\cos n\pi(e^{\pi}-e^{-\pi})}{\pi(n^2+1)}=\frac{2(-1)^n\sinh\pi}{\pi(n^2+1)}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\sin nxdx=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx-n\cos nx)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi(n^2+1)}\left[-ne^{\pi}\cos n\pi+ne^{-\pi}\cos n\pi\right]$$

$$= \frac{-n\cos n\pi\left(e^{\pi}-e^{-\pi}\right)}{\pi\left(n^2+1\right)} = \frac{2(-1)^{n+1} n\sinh\pi}{\pi\left(n^2-1\right)}$$

$$\therefore e^x = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n\sinh\pi}{\pi\left(1+n^2\right)}\cos nx + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1} n\sinh n\pi}{\pi\left(1+n^2\right)}\sin nx$$

$$\Rightarrow \frac{\pi}{2\sinh\pi}e^x = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{1+n^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1} n\sin nx}{1+n^2}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-1}(\cos nx - n\sin nx)$$

Hence $\frac{\pi}{2\sinh\pi}e^x = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)$

17. Find the Fourier series of $f(x)$ in $(-\pi,\pi)$,

where $f(x) = \begin{cases}\pi + x, & -\pi < x < 0\\ \pi - x, & 0 < x < \pi\end{cases}$ (April 2009)

Sol. Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

By Euler's formulae

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\,dx + \int_{0}^{\pi}(\pi-x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\pi x + \frac{x^2}{2}\right)_{-\pi}^{0} + \left(\pi x - \frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - \left(-\pi^2 + \frac{\pi^2}{2}\right) + \left(\pi^2 - \frac{\pi^2}{2}\right) - 0\right]$$

$$= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right] = \pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\cos nx\,dx - \int_{0}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\pi}(\pi-x)\cos nx\,dx\right]$$

In the first integral take

$x = -t \Rightarrow dx = -dt$

$x = -\pi \Rightarrow t = \pi$

$x = 0 \Rightarrow t = 0$

$\therefore$

$$\int_{-\pi}^{0}(\pi+x)\cos nx\,dx$$

$$= \int_{\pi}^{0}(\pi-t)\cos(-nt)(-dt)$$

$$= -\int_{\pi}^{0}(\pi-t)\cos nt\,dt$$

$$= \int_{0}^{\pi}(\pi-x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi-x)\frac{\sin nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{2}{n\pi}\left[0 - \frac{\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{2}{\pi n^2}\left[\cos n\pi - 1\right]$$

$$= \frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

$$= \begin{cases}0 & \text{if n is even}\\ \frac{4}{\pi n^2} & \text{if n is odd}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \int_{0}^{\pi}(\pi-x)\sin nx\,dx\right]$$

Now

$$\int_{-\pi}^{0}(\pi+x)\sin xdx$$

By the above substitutions

$$= \int_{\pi}^{0}(\pi-t)\sin(-nt)(-dt)$$

$$= \int_{\pi}^{0}(\pi-t)\sin nt\,dt$$

$$= -\int_{0}^{\pi}(\pi-x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\int_0^{\pi}(\pi-x)\sin nx\,dx + \int_0^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$= 0$$

$$\therefore f(x) = \frac{1}{2}(\pi) + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[1-(-1)^n\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \ldots\right]$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

Part - B

1

GROUPS

1. **State and prove Lagrange's Theorem for finite groups.**
(September 2013, April 2013, 2011, 2010, 2009)

Sol. Let $o(G) = n$.

Since corresponding to each element in G, we can define a right coset of H in G, the number of distinct right cosets of H in G is less than or equal to n.

Using the properties of equivalence classes we know

$G = Ha_1 \cup Ha_2 \cup \ldots \cup Ha_t$

Where t = no. of distinct right cosets of H in G.

$\Rightarrow o(G) = o(Ha_1) + o(Ha_2) + \ldots + o(Ha_t)$

(reminding ourselves that two right cosets are either equal or have no element in common).

$\Rightarrow o(G) = \underbrace{o(H) + o(H) + \ldots + o(H)}_{t \text{ times}}$

$\Rightarrow o(G) = t.o(H)$

or $o(H) | o(G)$

2. **Let H be a subgroup of a group G. If $x^2 \in H$ for all $x \in G$, then prove that H is a normal subgroup of G.** **(September 2013)**

Sol. Let $g \in G$ and $h \in H$ be any element

Now $ghg^{-1} = ghghh^{-1}g^{-2} = (gh)^2 h^{-1}g^{-2} \in H$

for $g, h \in G \Rightarrow gh \in G \quad \Rightarrow (gh)^2 \in H$

Also $g^{-1} \in G \Rightarrow g^{-2} \in H$

Also $h^{-1} \in H$

Hence $(gh)^2 h^{-1} g^{-2} \in H$

$\Rightarrow ghg^{-1} \in H \quad \forall g \in G,\ h \in H$

$\therefore$ H is a normal subgroup of G.

3. Prove that any finite cyclic group of order n is isomorphic to quotient group $Z|<n>$. **(September 2013, 2013)**

Sol. Let G = <a> be of order n

Define $f: Z \to G$, s.t.,

$f(m) = a^m$

then f is clearly well defined onto map.

Since $f(m+k) = a^{m+k} = a^m.a^k = f(m).f(k)$

f is a homomorphism and therefore, by Fundamental theorem $G \cong \frac{Z}{\text{Ker} f}$

We show $\text{Ker} f = N = <n>$

Now $m \in \text{Ker} f \Leftrightarrow f(m) = e$

$\Leftrightarrow a^m = e$

$\Leftrightarrow o(a)|m$

$\Leftrightarrow n|m$

$\Leftrightarrow m \in <n>$

Hence $G \cong \frac{Z}{<n>}$.

4. Prove that Alternating group A_4 has no subgroup of order six. **(September 2013, 2012, 2010)**

Sol. Consider the alternating group A_4.

$o(A_4) = \frac{o(S_4)}{2} = \frac{\lfloor 4}{2} = 12$

We show although 6 | 12, A_4 has no subgroup of order 6. Suppose H is a subgroup of A_4 and o(H) = 6.

Now the number of distinct 3-cycles in S_4 is

$\frac{1}{3}.\frac{4!}{(4-3)!} = \frac{4.3.2.1}{3.1} = 8.$

Again, as each 3-cycle will be even permutation all these 3-cycles are in A_4.

Obviously then, at least one 3-cycle, say σ, does not belong to H (o(H) = 6).

Now $\sigma \notin H \Rightarrow \sigma^2 \notin H$, because if $\sigma^2 \in H$

then $\sigma^4 \in H$

$\Rightarrow \sigma \in H$

As $\sigma^3 = I$ as $o(\sigma) = 3$.

Let $K = <\sigma> = \{I, \sigma, \sigma^2\}$ then $o(K) = 3$

and $H \cap K = \{I\}$ $(\sigma, \sigma^2 \notin H)$

$\Rightarrow o(HK) = \frac{o(H).o(K)}{o(H \cap K)} = \frac{6.3}{1} = 18$, not possible as $HK \subseteq A_4$ and $o(A_4) = 12$.

5. Prove that $H \neq \phi \subset G$ is a subgroup of G iff $\forall a, b \in H \Rightarrow ab^{-1} \in H$. **(April 2013)**

Sol. If H is a subgroup of G then, a, b ∈ H $\Rightarrow ab^{-1} \in$ H (by using definition).

Conversely, let the given condition hold in H.

Associativity holds in H as $H \subset G$ and G is a group.

Let a ∈ H be any element (H ≠ ϕ)

then a, a ∈ H $\Rightarrow aa^{-1} \in H \Rightarrow$ e ∈ H.

So H has identity.

Again, for any a ∈ H, as e ∈ H

$ea^{-1} \in H \Rightarrow a^{-1} \in H$

i.e., H has inverse of each element.

Finally, for any $a, b \in H$,

$a, b^{-1} \in H$

$\Rightarrow a(b^{-1})^{-1} . \in H \Rightarrow ab \in H$

i.e., H is closed under multiplication.

Hence H forms a group and therefore a subgroup of G.

6. If G = <a> is a cyclic group generated by 'a'. Prove that o(G) = o(a). **(April 2013)**

Sol. Let $G = <a>$ i.e., G is a cyclic group generated by a.

Case (i): o(a) is finite, say n, then n is the least +ve integer s.t., $a^n = e$.

Consider the elements $a^0 = e, a, a^2, \ldots\ldots, a^{n-1}$

These are all elements of G and are n in number.

Suppose any two of the above elements are equal

say $a^i = a^j$ with $i > j$

then $a^i.a^{-j} = e \Rightarrow a^{i-j} = e$

But $0 < i - j \leq n - 1 < n$, thus ∃ a +ve integer $i - j$, s.t., $a^{i-j} = e$ and $i - j < n$ which is a contradiction to the fact that o(a) = n.

Thus no two of the above n elements can be equal i.e., G contains at least n elements. We show it does not contain any other element. Let x ∈ G be any element. Since G is cyclic, generated by a,x will be some power of a.

Let $x = a^m$

By division algorithm, we can write

$m = nq + r$ where $0 \le r < n$

Now $a^m = a^{nq+r} = (a^n)^q . a^r = e^q . a^r = a^r$

$\Rightarrow x = a^r$ where $0 \le r < n$

i.e., x is one of $a^0 = e, a, a^2, ..., a^{n-1}$

or G contains precisely n elements

$\Rightarrow o(G) = n = o(a)$

Case (ii): o(a) is infinite.

In this case no two powers of a can be equal as if $a^n = a^m \, (n > m)$ then $a^{n-m} = e$, i.e., it is possible to find a +ve integer n – m s.t., $a^{n-m} = e$ meaning thereby that a has finite order.

Hence no two power of a can be equal. In other words G would contain infinite number of elements.

7. If f be homomorphism of group G onto G' with kernel K then $\frac{G}{K} \cong G'$.

(April 2013, September 2010, April 2010, September 2009)

Sol. Define a map: $\varphi : \frac{G}{K} \to G'$,

s.t., $\varphi(Ka) = f(a), a \in G$

We show φ is an isomorphism.

That φ is well defined follows by

$Ka = Kb$

$\Rightarrow ab^{-1} \in K = \text{Ker} f$

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow f(a)(f(b))^{-1} = e'$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \varphi(Ka) = \varphi(Kb)$

By retracing the steps backwards, we will prove that φ is 1-1.

Again as $\varphi(KaKb) = \varphi(Kab) = f(ab) = f(a)f(b) = \varphi(Ka)\varphi(Kb)$

We find φ is a homomorphism.

To check that φ is onto, let $g' \in G'$ be any element. Since $f : G \to G'$ is onto, ∃ g ∈ G, s.t.,

$f(g) = g'$

Now $\varphi(Kg) = f(g) = g'$

Showing thereby that Kg is the required pre-image of g' under φ.

Hence φ is an isomorphism.

$\therefore \frac{G}{K} \cong G'$

8. Prove that a finite semi-group in which both the cancellation laws hold is a group.

(September 2012, September 2011)

Sol. Let S be a finite semi-group in which both cancellation laws hold.

Let $S = \{a_1, a_2, a_3, ..., a_n\}$ (1)

Consider any $a_l \in S$, then

$a_1a_l, a_2a_l, a_3a_l, ..., a_na_l$ (2)

are all distinct, since for any i and j $(1 \le i, j \le n)$

$a_ia_l = a_ja_l \Rightarrow a_i = a_j$ (right cancellation law).

Hence elements in (2) are all the elements in (1) in possibly some other order. So given any $a_i \in S \; \exists a_j \in S$ such that $a_i = a_ja_l$ (3)

In particular there exists some $a_k \in S$ such that $a_l = a_ka_l$.

Then $a_ia_l = a_i(a_ka_l) = (a_ia_k)a_l$

So by right cancellation law in S, $a_i = a_ia_k$ for all i.

Similarly considering the elements $a_la_1, a_la_2, ..., a_la_n$ and using left cancellation law we can find an element a_t such that $a_ta_i = a_i$ for all i.

Then we get $a_k = a_ta_k = a_t$

and $a_ka_i = a_i = a_ia_k$ for all i = 1, 2, ... , n.

Hence a_k is the identity element in S.

Let us write e for a_k. Taking $i = k$ in (3) we can say that there exists a_m such that $e = a_k = a_ma_l$. In a similar fashion, by considering the elements $a_la_1, a_la_2, ..., a_la_n$ we can find an element a_u such that $a_la_u = e$.

Then $a_u = ea_u = (a_ma_l)a_u = a_m(a_la_u) = a_me = a_m$

and $a_ma_l = e = a_la_m$. So $a_l^{-1} = a_m$.

Consequently every element in S has its inverse in S. S is such that for all a, b, c ∈ S

(i) $a(bc) = (ab)c$ (S is a semi-group).

(ii) There exists e ∈ S such that

$ae = a = ea \ \forall a \in S$.

(iii) Given $a \in S$, $\exists a^{-1} \in S$ such that $aa^{-1} = e = a^{-1}a$.

Hence S is a group.

This proves the theorem.

9. If H is a subgroup of G of index 2 in G. Then H is a normal subgroup of G. **(September 2012, April 2010)**

Sol. Let H be a subgroup of G, with index 2 then number of distinct right (left) cosets of H in G is 2 and also then G is union of these two right (left) cosets.

Let $g \in G$ be arbitrary.

Case (i): $g \in H$, then $Hg = gH (= H)$

Hence H is normal.

Case (ii): $g \notin H$ then $gH \neq H, Hg \neq H$.

Thus Hg and H = He are the two distinct right cosets of H in G and

$G = Hg \cup H$

Similarly, $G = gH \cup H$

$\Rightarrow \ Hg \cup H = gH \cup H$

$\Rightarrow \ Hg = gH$ (as $Hg \cap H = gH \cap H = \varphi$)

$\Rightarrow$ H is normal in G.

10. Let G be a set with binary operation which is associative. Assume that for all elements a and b in G, the equations ax = b and ya = b have unique solution in G. Then prove that G is a group. **(April 2012)**

Sol. To show G is a group, we need to prove existence of identity and inverse for each element.

Let $a \in G$ be any element.

Given, the equations $ax = a$

$ya = a$

have solutions in G.

Let $x = e$ and $y = f$ be the solutions.

Thus $\exists \ e, f \in G$, s.t., $ae = a$

$fa = a$

Let now $b \in G$ be any element then again $\exists$ some x, y in G s.t.,

$ax = b$

$ya = b$

Now $\quad ax = b \Rightarrow f.(a.x) = f.b$

$\Rightarrow (f.a).x = f.b$

$\Rightarrow a.x = f.b$

$\Rightarrow b = f.b$

Again $\quad y.a = b \Rightarrow (y.a).e = b.e$

$\Rightarrow y.(a.e) = b.e$

$\Rightarrow y.a = be$

$\Rightarrow b = be$

thus we have $\quad b = fb \quad$ (i)

and $\quad b = be \quad$ (ii)

for any $\quad b \in G$

Putting b = e in (i) and $b = f$ in (ii) we get

$e = fe$

and $\quad f = fe$

$\Rightarrow e = f$

Hence $\quad ae = a = fa = ea$

i.e., $\exists \ e \in G$, s.t., $ae = ea = a$

$\Rightarrow$ e is identity.

Again, for any $a \in G$, and (the identity) $e \in G$, the equations $ax = e$ and $ya = e$ have solutions.

Let the solutions be $x = a_1$, and $y = a_2$

Then $\quad aa_1 = e, \ a_2a = e$

Now $\quad a_1 = ea_1 = (a_2a)a_1 = a_2(aa_1) = a_2e = a_2$.

Hence $\quad aa_1 = e = a_1a$ for any $a \in G$

i.e., for any $a \in G, \exists$ some $a_1 \in G$ satisfying the above relations $\Rightarrow$ a has an inverse. Thus each element has inverse and, by definition, G forms a group.

11. Let G be a group and H and K subgroups of finite indices in G. Show that H ∩ K is also of finite index. If (G : H) = m and (G : K) = n with (m, n) = 1, show that (G : H ∩ K) = mn. **(April 2012)**

Sol. First we will show that if G is finite group H,K subgroups of G such that K ⊂H

Then $[G:K] = [G:H][H:K]$

Let $[G:H] = m \ \ [H:K] = n$

$\Rightarrow$ G = ⋃ aH a disjoint union of m cosets

H = ⋃ b K a disjoint union of n cosets

Then G = $\bigcup a\,b\,K$ is disjoint union of mn cosets

$\Rightarrow [G:K] = mn = [G:H][H:K]$

Using the above result

$[G:H\cap K] = [G:H][H:H\cap K]$

It is given $[G:K] = n$ and $[G:H] = m$ with (m, n) = 1

$[G:K]$ is finite $\Rightarrow [H:H\cap K] \le [G:K]$

Hence $[G:H\cap K] \le mn$

But

$[G:H]$ divides $[G:H\cap K]$ and $[G:K]$ divides $[G:H\cap K]$

So their product mn divides $[G:H\cap K]$ $\quad\because$ (m, n) = 1

$\Rightarrow mn \le [G:H\cap K]$

Hence $[G:H\cap K] = mn$

$= [G:H][G:K]$

12. Define center Z(G) of a group G. Show that if $\dfrac{G}{Z(G)}$ is cyclic, then G is Abelian.

(April 2012)

Sol. **Center of group:** $Z(G) = \{z \in G : zx = xz \;\; \forall x \in G\}$

Let us write Z(G) = N. Then $\dfrac{G}{N}$ is cyclic. Suppose it is generated by Ng.

Let $a, b \in G$ be any two elements,

then $Na, Nb \in \dfrac{G}{N}$

$\Rightarrow Na = (Ng)^n, Nb = (Ng)^m$ for some n, m

$\Rightarrow Na = \underbrace{Ng.Ng........Ng}_{n \text{ times}} = Ng^n$

Similarly, $Nb = Ng^m$

$\Rightarrow ag^{-n} \in N, bg^{-m} \in N$

$\Rightarrow ag^{-n} = x, bg^{-m} = y$ for some $x, y \in N$

$\Rightarrow a = xg^n, b = yg^m$

$\Rightarrow ab = (xg^n)(yg^m) = x(g^n y)g^m$

$= x(yg^n)g^m$ as $y \in N = Z(G)$

$= xyg^n g^m$

$= xyg^{n+m}$

Similarly, $ba = (yg^m)(xg^n) = y(g^m x)g^n = y(xg^m)g^n = (yx)g^{m+n}$

$\Rightarrow ab = ba$ as $xy = yx$ as $x, y \in Z(G)$

Showing that G is abelian.

13. Prove that any non-cyclic group of order 4 is isomorphic to the Klein 4-group.

(April 2012)

Sol. Any non cyclic abelian group of order 4 is of the type $G = \{e, a, b, ab\}$

Here every element (except e) will be of order 2

$[\because$ if $x \in G$ then $o(x) | o(G)$

$\Rightarrow o(x) = 1, 2,$ or 4

But $o(x) = o(G) = 4$ implies G is cyclic which is not so$]$

This non cyclic group of order 4 is isomorphic to Klein's-4 group

$K_4 = \{I, (12)(34), (13)(24), (14)(23)\}$

and the isomorphism is:

$e \to I$

$a \to (12)(34)$

$b \to (13)(24)$

$ab \to (14)(23)$

Hence every non cyclic abelian group of order 4 is isomorphic to Klein's four group.

14. Let S_n be the symmetric group on n symbols and A_n be subset of S_n consisting of all even permutations. Prove that A_n is a normal subgroup of S_n of index 2.

(April 2012, September 2009)

Sol. Since identity permutation is even, A_n is a non empty subset of S_n.

Again, $f, g \in A_n \Rightarrow f, g$ are even permutations

$\Rightarrow f, g^{-1}$ are even permutations

$\Rightarrow fog^{-1}$ is even

$\Rightarrow fog^{-1} \in A_n$

or that A_n is a subgroup of S_n.

If $f \in A_n$ and $g \in S_n$ be any members then $g^{-1} o\, fog$ will be even permutation, showing that $g^{-1} o\, fog \in A_n$ or that A_n is a normal subgroup of S_n.

Let $G=\{1,-1\}$ be the group under multiplication.

Define a map $\varphi: S_n \to G$, s.t.,

$$\varphi(f)=\begin{cases}1 & \text{if f is even permutation}\\ -1 & \text{if f is odd permutation}\end{cases}$$

then φ is an onto mapping as $S_n\ (n\geq 2)$ must contain even as well as odd permutations. (Identity permutation and (12) will be in S_n).

To show that φ is a homomorphism

Let $f,g\in S_n$ be any members.

Case (i): Both f, g are even, then fog is even

$\varphi(fog)=1=1.1=\varphi(f)\varphi(g)$

Case (ii): Both f, g are odd, then fog is even

$\varphi(fog)=1=(-1)(-1)=\varphi(f)\varphi(g)$

Case (iii): One of f, g is odd, other even.

Suppose f is odd and g is even, then fog is odd

$\varphi(fog)=-1=(-1)(1)=\varphi(f)\varphi(g)$

hence φ is an onto homomorphism and thus by Fundamental theorem of homomorphism

$$G\cong\frac{S_n}{\text{Ker }\varphi}$$

Since $f\in \text{Ker }\varphi \Leftrightarrow \varphi(f)=1$

$\Leftrightarrow$ f is even $\Leftrightarrow f\in A_n$

We have $\text{Ker }\varphi = A_n$

or that $G\cong\frac{S_n}{A_n}$

But $o(G)=2 \Rightarrow o\left(\frac{S_n}{A_n}\right)=2$

$\Rightarrow \frac{o(S_n)}{o(A_n)}=2$

$\Rightarrow \frac{o(S_n)}{2}=o(A_n)$

Thus index of A_n in S_n is 2.

15. For any subgroup H of a finite group G, prove that o(G) = o(H) [G:H].

(September 2011)

Sol. Let $o(G)=n$.

Since corresponding to each element in G, we can define a right coset of H in G, the number of distinct right cosets of H in G is less than or equal to n.

Using the properties of equivalence classes we know

$G=Ha_1\cup Ha_2\cup....\cup Ha_t$

where t = no. of distinct right cosets of H in G.

$\Rightarrow o(G)=o(Ha_1)+o(Ha_2)+....+o(Ha_t)$

(reminding ourselves that two right cosets are either equal or have no element in common).

$\Rightarrow o(G)=\underbrace{o(H)+o(H)+....+o(H)}_{t\text{ times}}$

$\Rightarrow o(G)=t.o(H)$

or that $o(H)\,|\,o(G)$

Let G be a group and H, a subgroup of G. Then index of H in G is the number of distinct right (left) cosets of H in G. It is denoted by $i_G(H)$ or $[G:H]$.

If G is a finite group, then $i_G(H)=[G:H]=\frac{o(G)}{o(H)}$.

$\Rightarrow o(G)=o(H)[G:H]$

16. Prove that if H and K are finite subgroups of a group G then $o(HK)=\frac{o(H)o(K)}{o(H\cap K)}$

(September 2011, September 2009)

Sol. Let $D=H\cap K$ and D is a subgroup of K and as in the proof of Lagrange's theorem, $\exists$ a decomposition of K into disjoint right cosets of D in K and

$K=Dk_1\cup Dk_2\cup.....\cup Dk_t$

and also $t=\frac{o(K)}{o(D)}$

Again, $HK=\bigcup_{i=1}^{t}Hk_i=Hk_1\cup Hk_2\cup....\cup Hk_t$

Now no two of $Hk_1, Hk_2, ..., Hk_t$ can be equal as if $Hk_i=Hk_j$ for i, j

Then $k_ik_j^{-1}\in H\Rightarrow k_ik_j^{-1}\in H\cap K\Rightarrow k_ik_j^{-1}\in D\Rightarrow Dk_i=Dk_j$

which is not true.

Hence $o(HK)=o(Hk_1)+(Hk_2)+....+o(Hk_t)$

$$= \underbrace{o(H)+o(H)+....+o(H)}_{t \text{ times}}$$

$$= t.o(H)$$

$$= \frac{o(H).o(K)}{o(H \cap K)}$$

which proves the result.

17. If H and K are two normal subgroups of G such that $H \subseteq K$, then prove that $\frac{K}{H}$ is a normal subgroup of $\frac{G}{H}$ and $\frac{G}{K} \cong \frac{G/H}{K/H}$. **(September 2011)**

Sol. Define a map $f : \frac{G}{H} \to \frac{G}{K}$

s.t., f (Ha) = Ka

f is well defined as

$Ha = Hb$

$\Rightarrow ab^{-1} \in H \subseteq K \quad \Rightarrow \quad ab^{-1} \in K$

$\Rightarrow Ka = Kb$

$\Rightarrow f(Ha) = f(Hb)$

f is a homomorphism as

$f(HaHb) = f(Hab) = Kab = KaKb = f(Ha)f(Hb)$.

Ontoness of f is obvious.

Using Fundamental theorem of group homomorphism we can say

$\frac{G}{K} \cong \frac{G/H}{\text{Ker} f}$

We claim Ker $f = \frac{K}{H}$

A member of Ker f will be some member of $\frac{G}{H}$.

Now $Ha \in \text{Ker} f \Leftrightarrow f(Ha) = K$ (identity of G/K)

$\Leftrightarrow Ka = K$

$\Leftrightarrow a \in K$

$\Leftrightarrow Ha \in \frac{K}{H}$

Hence we find $\frac{G}{K} \cong \frac{G/H}{K/H}$

18. Prove that there are only two groups of order six, one is cyclic and other is isomorphic to S_3. **(September 2011, April 2011)**

Sol. Let G be a group of order 6. Since there are five non-identity elements and $x \leftrightarrow x^{-1}$ is a 1 – 1 correspondence between the elements and their inverses, there exists $a(\neq e) \in G$ such that a is its own inverse i.e. $o(a) = 2$.

Let G be Abelian and H be the sub-group <a>. Now $o(G/H) = 3 \Rightarrow G/H$ is cyclic.

Let bH be a generator of G/H. Since $o(bH) | o(b)$, we get $o(b) = 3$ or 6.

If $o(b) = 6$ then trivially G is cyclic.

If $o(b) = 3$ then $o(ab) = 6$ as $ab = ba$ and $o(a)$ and $o(b)$ are coprime.

Hence again G is cyclic group generated by ab.

Suppose now G is not cyclic then G cannot be Abelian. Then all non-identity elements of G cannot be of order 2.

Thus there exists c in G such that $o(c) = 3$ and $ac \neq ca$.

Let $H = \{e, c, c\}$. As $[G : H] = 2$, H is a normal subgroup. Consequently $a^{-1}ca \in H$ So that $a^{-1}ca = c$ or c^2; but as $ac \neq ca$, we get $a^{-1}ca = c^2$; i.e. $ca = ac^2$.

Now a $\notin H$ so we get $G = H \cup aH = \{e, c, c^2, a, ac, ac^2\}$.

$\therefore$ the mapping $f : G \to S_3$ given by $f(e) = I, f(a) = (12), \; f(c) = (123),$ $f(c^2) = (132), f(ac) = (23)$ and $f(ac^2) = (13)$ is an isomorphism.

Hence the result follows.

19. Let H and K be any two subgroups of a group G. Prove that: **(April 2011)**

(i) $H \cap K$ is a subgroup of G.

(ii) $H \cap K$ is normal in K if H is normal in G.

Sol. (i) Let H and K be any two subgroups of G

then $H \cap K \neq \phi$ atleast e is common to H and K

Let a, b$\in H \cap K$

$\Rightarrow$ a $\in$ H and b $\in$ H

and a $\in$ K and b $\in$ K

Since H, K are subgroups

a $\in$ H, b $\in$ H $\Rightarrow ab^{-1} \in$ H

a $\in$ K, b $\in$ K $\Rightarrow ab^{-1} \in$ K

$\Rightarrow ab^{-1} \in H \cap K$

Hence $H \cap K$ is subgroup of G.

(ii) H is normal in G

and $H \cap K$ is subgroup of G

Also $H \cap K \subseteq$ K

Therefore $H \cap K$ is a subgroup of K

Next to show $H \cap K$ is normal subgroup of K

Let x ∈ K and a ∈ $H \cap K$

a ∈ $H \cap K \Rightarrow$ a ∈ H and a ∈ K

Since H is normal subgroup of G therefore $xax^{-1} \in H$

Also K is subgroup of G ⇒ x ∈ K, a ∈ K,

$\Rightarrow xax^{-1} \in K$

Hence $xax^{-1} \in H \cap K$

⇒ x ∈ K, a ∈ $H \cap K \Rightarrow xax^{-1} \in H \cap K$

Thus $H \cap K$ is normal subgroup of K.

20. Let G be a group of order 2n, where n is odd. Show that G has a normal subgroup of order n. **(April 2011)**

Sol. G possesses at least one element of order 2. Otherwise we can partition G into union of disjoint pairs $\{a, a^{-1}\}, \{b, b^{-1}\}, ...$ etc. and a singleton $\{e\}$, where e is the identity of G. This will in turn imply that number of elements in G is odd! So, let a ∈ G be of order 2. Consider now T_a where $T_a(x) = ax$ for all x∈G.

T_a is a permutation on the set G. Since $T_a^2(x) = T_a[T_a(x)] = a(ax) = a^2x = ex = x$, each T_a-orbit of G consists of two elements and number of distinct orbit of $T_a = \frac{2n}{2} = n$.

As each orbit corresponds to a cycle, T_a has n cycles each of length 2.

Let $\sigma_1, \sigma_2, ..., \sigma_n$ be all the n 2-cycles (i.e. transpositions) of T_a. So $T_a = \sigma_1, \sigma_2, ..., \sigma_n$. As n is odd, T_a is an odd permutation in A(G).

Let $K = \{T_a \mid g \in G\}$, K is a subgroup of $A(G)$ and $G \cong K$ under the mapping $f(g) = T_a$. As $T_a \in K$, K contains an odd permutation.

So we get that K contains a normal subgroup N of index 2.

Thus G contains a normal subgroup $H = f^{-1}(N)$ of index 2. But $o(H) = \frac{o(G)}{i(H)} = \frac{2n}{2} = n$. Hence G has a normal subgroup of order n.

21. Show that the equation $x^2ax = a^{-1}$ is solvable for x in a group if and only if a is the cube of some element in G. **(September 2010)**

Sol. Suppose $x^2ax = a^{-1}$ is solvable in G.

Then there exists an element c∈G such that $c^2ac = a^{-1}$

$ccac = a^{-1}$

$$\Rightarrow c(ca)ca = a^{-1}a$$

$$\Rightarrow c(ca)(ca) = e$$

$$\Rightarrow (ca)(ca) = c^{-1}$$

$$\Rightarrow (ca)(ca)c = c^{-1}c$$

$$\Rightarrow (ca)(ca)c = e$$

$$\Rightarrow (ca)(ca)ca = ea$$

$$\Rightarrow (ca)^3 = a$$

⇒ a is cube of some element ca∈G

Conversely, Let $a = b^3$ for some b∈G then $x = b^{-2}$ is solution of $x^2ax = a^{-1}$

If $x = b^{-2}$ and $a = b^3$

then $x^2ax = b^{-4}b^3b^{-2} = b^{-3} = (b^3)^{-1} = a^{-1}$

Hence $x = b^{-2}$ is solution of $x^2ax = a^{-1}$

22. Prove that a group of order n is cyclic if and only if it has an element of order n. **(September 2010)**

Sol. Suppose G is a finite group of order n.

Let a∈G and let n be the order of a.

If H is the cyclic subgroup of G generated by a i.e., if $H = \{a^r : r \in I\}$, then the order of H is n because the order of the generator a of H is n.

Thus H is a cyclic subgroup of G and the order of H is equal to the order of G. Hence H = G and therefore G itself is a cyclic group and a is a generator of G.

23. Define index of a subgroup. If H ⊆ K are two subgroups of a finite group G, then show that [G : H] = [G : K] [K : H]. **(September 2010)**

Sol. If H is subgroup of a finite group G then index of H in G = the number of distinct right (or left) cosets of H in G

$$[G:H] = \frac{o(G)}{o(H)}$$

Since H ⊆ K be two subgroups of a group G therefore H is also a subgroup of K

Also H is subgroup of finite group G.

Therefore

$$[G:H]=\frac{o(G)}{o(H)}=\frac{o(G)}{o(K)}\cdot\frac{o(K)}{o(H)}$$

$$=[G:K][K:H]$$

Hence $[G:H]=[G:K][K:H]$

24. Prove that a subgroup of a cyclic group is cyclic. **(April 2010)**

Sol. Let G = <a> and let H be a subgroup of G.

If $H=\{e\}$, there is nothing to prove. Let $H \neq \{e\}$. Members of H will be powers of a.

Let m be the least +ve integer s.t., $a^m \in H$. We claim $H = \langle a^m \rangle$.

Let $x \in H$ be any element. Then $x = a^k$ for some k. By division algorithm, $k = mq + r$ where $0 \le r < m$

$\Rightarrow r = k - mq$

$\Rightarrow a^r = a^{k-mq} = a^k . a^{-mq} = x.(a^m)^{-q} \in H$

But m is the least +ve integer s.t., $a^m \in H$, meaning thereby that r = 0.

Thus $k = mq$

or that $x = a^k = (a^m)^q$

i.e., any member of H is a power of a^m.

25. Define class equation of a finite group and prove that if $o(G) = p^n$ (where p is a prime number and $n \ge 1$) then centre $Z = \{e\}$. **(April 2010, April 2009)**

Sol. **Class Equation:** Let G be a finite group and Z be the centre of G. Then the class equation of G can be written as

$$o(G)=o(Z)+\sum_{a\notin Z}\frac{o(G)}{o[N(a)]},$$

where the summation runs over one element a in each conjugate class containing more than one element.

By the class equation of G, we have

$$o(G)=o(Z)+\sum_{a\notin Z}\frac{o(G)}{o[N(a)]}, \qquad (1)$$

where the summation runs over one element a in each conjugate class containing more than one element.

Now $\forall a \in G, N(a)$ is a subgroup of G. Therefore by Lagrange's theorem, $o[N(a)]$ is a divisor of $o(G)$. Also $a \notin Z \Rightarrow N(a) \neq G \Rightarrow o[N(a)] < o(G)$.

Therefore if $a \notin Z$, then $o[N(a)]$ must be of the form P^{n_a} where n_a is some integer such that $1 \le n_a < n$.

Suppose there are exactly z elements in Z i.e., let $o(Z) = z$. Then the class equation (1) gives

$P^n = z + \frac{P^n}{P^{n_a}}$ where each n_a is some integer such that

$1 \le n_a < n$.

$$\therefore z = P^n - \frac{P^n}{P^{n_a}} \qquad (2)$$

where n_a's are some positive integers each being less than n.

Now $p \mid p^n$. Also p divides each term in the summation of the right hand side of (2) because each $n_a < n$. Thus we see that p is a divisor of the right hand side of (2). Therefore p is a divisor of z.

Now e∈Z. Therefore $z \neq 0$. Therefore z is a positive integer divisible by the prime p. Therefore z > 1. Hence Z must contain an element besides e. Therefore $Z \neq \{e\}$.

26. Let G be the set of all real 2 × 2 matrices $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ where $a \neq 0$. Prove that G is an abelian group under the matrix multiplication. What is order of this group? **(September 2009)**

Sol. $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : \text{a is non zero real} \right\}$

Let $\alpha = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \beta = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \in G$

$$\alpha\beta = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} ba & 0 \\ 0 & a^{-1}b^{-1} \end{bmatrix}$$

$$\beta\alpha = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} ba & 0 \\ 0 & b^{-1}a^{-1} \end{bmatrix}$$

$ab = ba$ and $a^{-1}b^{-1} = b^{-1}a^{-1}$ ($\because$ a, b are non zero reals)

Hence $\alpha\beta = \beta\alpha$

$\Rightarrow$ G is abelian group under multiplication

Order of this group is infinite because there are infinite non-zero reals. Hence there are infinite elements in group G.

$\therefore$ o(G) = Infinite

27. Let G be a finite group and let $a \in G$ be an element of order n. Then prove that $a \in G$ iff n is a divisor of m. **(September 2009)**

Sol. Let $o(a) = n$, they by definition n is the least +ve integer s.t., $a^n = e$.

Suppose $a^m = e$ for some m.

By division algorithm, $m = nq + r$, where $0 \le r < n$

$a^m = a^{nq+r}$

$\Rightarrow e = a^{nq}.a' = (a^n)^q.a^r = e^q.a^r = a^r$

where $0 \le r < n$

Since n is the such least +ve integer, we must have $r = 0$

i.e., $m = nq$ or that $n \mid m$.

28. Prove that an infinite cyclic group has precisely two generators. (September 2009)

Sol. Let G = <a> be an infinite cyclic group.

If a is a generator of G then so would be a^{-1}.

Let now b be any generator of G,

then as $b \in G$ and a generates G, we get $b = a^n$ for some integer n

again as $a \in G$ and b generates G, we get $a = b^m$ for some integer m

$\Rightarrow a = b^m = (a^n)^m = a^{nm}$

$\Rightarrow a^{nm-1} = e \Rightarrow o(a)$ is finite and $\le nm - 1$

Since $o(G) = o(a)$ is infinite, the above can hold only if

$nm - 1 = 0 \Rightarrow nm = 1$

$\Rightarrow m = \frac{1}{n}$ or $n = \pm 1$ as m, n are integers.

i.e., $b = a$ or a^{-1}

In other words, a and a^{-1} are precisely the generators of G.

29. Define order of an element of a group G.

Let $a, b \in G$ be non-identity elements with $o(a) = 5$ and $aba^{-1} = b^2$, find $o(b)$.

Sol. Definition: Let G be a group and $a \in G$. If n is the least +ve integer such that $a^n = e$ then n is said to be the order of the element a and write $o(a) = n$.

We have $b^2 = aba^{-1}$

$\Rightarrow b^4 = (aba^{-1})(aba^{-1})$

$= ab(a^{-1}a)ba^{-1} = abba^{-1} = ab^2a^{-1}$

$= a(aba^{-1})a^{-1} = a^2ba^{-2}$

$\Rightarrow b^8 = (a^2ba^{-2})(a^2ba^{-2})$

$= a^2b(a^{-2}a^2)ba^{-2} = a^2bba^{-2} = a^2b^2a^{-2}$

$= a^2(aba^{-1})a^{-2} = a^3ba^{-3}$

$\Rightarrow b^{16} = (a^3ba^{-3})(a^3ba^{-3})$

$= a^3b(a^{-3}a^3)ba^{-3} = a^3bba^{-3} = a^3b^2a^{-3}$

$= a^3(aba^{-1})a^{-3} = a^4ba^{-4}$

$\Rightarrow b^{32} = (a^4ba^{-4})(a^4ba^{-4})$

$= a^4b(a^{-4}a^4)ba^{-4} = a^4bba^{-4} = a^4b^2a^{-4}$

$= a^4(aba^{-1})a^{-4} = a^5ba^{-5}$

$= b$ $\qquad (\because a^5 = a^{-5} = e)$

$\Rightarrow b^{32} = b$

$\Rightarrow b^{31} = e$

$\Rightarrow o(b) = 31$

30. Let G and G' be two groups. If $f : G \to G'$ is homomorphism, show that the Kernel of f is a normal subgroup of G. **(April 2009)**

Sol. Since $f(e) = e', e \in$ Ker$f$, thus $Ker f \ne \phi$ Again,

$x, y \in$ Ker$f \Rightarrow f(x) = e'$

$f(y) = e'$

Now $f(xy^{-1}) = f(x)f(y^{-1}) = f(x)(f(y))^{-1} = e'.e'^{-1} = e'$

$\Rightarrow xy^{-1}$ Ker$f$

Hence it is a subgroup of G.

Again, for any $g \in G, x \in$ Ker$f$

$f(g^{-1}xg) = f(g^{-1})f(x)f(g)$

$= (f(g))^{-1} f(x)f(g) = (f(g))^{-1} e' f(g)$

$= (f(g))^{-1} f(g) = e'$

$\Rightarrow g^{-1}xg \in \text{Ker} f$

or that it is a normal subgroup of G.

31. Prove that $o(G) = p^2$ where p is a prime number, then G is an abelian group. Is group of order 2 abelian?

Sol. We will show that centre Z of G is equal to G itself. Then obviously G will be abelian group.

Since p is prime number $Z \neq \{e\}$

Hence o (Z) > 1 but Z is subgroup of G

$\therefore o(Z) = p$ or p^2

If $o(Z) = p^2$ then $Z = G \Rightarrow G$ is abelian

Let $o(Z) = p$ and $o(Z) < o(G)$ $\quad \therefore p < p^2$

then there exist an element in G which is not in Z.

Let $a \in G$ and $a \notin Z$

Also $N(a)$ normalizer of a is subgroup of G and $a \in N(a)$.

Also $x \in Z \Rightarrow xa = ax \Rightarrow x \in N(a)$

Thus $Z \subseteq N(a)$

Since $a \notin Z$

$\therefore$ number of elements in $N(a) > p$

i.e. $o[N(a)] > p$

But order of N(a) must be divisor of p^2.

$\therefore o[N(a)] = p^2 \Rightarrow N(a) = G$

$\Rightarrow a \in Z$ a contradiction

Hence $o(Z) = p^2 \Rightarrow Z = G \Rightarrow G$ is abelian

Yes, group of order 2 is abelian because every group of prime order is abelian.

GS 2·5

2

RINGS

1. If an ideal M of a commutative ring with unity is a maximal ideal, then Prove that R | M is a field.

(September 2013, April 2013, Sep, 2012, Sept. 2011, April 2009)

Sol. Let M be maximal ideal of R. Since R is commutative ring with unity, $\frac{R}{M}$ is also a commutative ring with unity. Thus all that we need prove is that non zero elements of $\frac{R}{M}$ have multiplicative inverse.

Let $x + M \in \frac{R}{M}$ be any non zero element.

then $x + M \neq M \Rightarrow x \notin M$

Let $xR = \{xr \mid r \in R\}$

It is easy to verify that xR is an ideal of R. Since sum of two ideals is an ideal, $M + xR$ will be an ideal of R.

Again as $x = 0 + x.1 \in M + xR$ and $x \notin M$ we find

$M \subset M + xR \subseteq R$

But M maximal $\Rightarrow M + xR = R$

Thus $1 \in R \Rightarrow 1 \in M + xR$

$\Rightarrow 1 = m + xr$ for some $m \in M, r \in R$

$\Rightarrow 1 + M = (m + xr) + M$

$= (m + M) + (xr + M) = xr + M$

$= (x + M)(r + M)$

$\Rightarrow (r + M)$ is multiplicative inverse of $x + M$

Hence $\frac{R}{M}$ is a field.

Conversely, let $\frac{R}{M}$ be a field.

Let I be any ideal of R s.t., $M \subset I \subseteq R$

then $\exists$ some $a \in I$, s.t., $a \notin M$

Now $a \notin M \Rightarrow a + M \neq M \Rightarrow a + M$ is a non zero element of $\frac{R}{M}$, which being a field\ means $a+M$ has multiplicative inverse. Let $b+M$ be its inverse. Then

$(a+M)(b+M) = 1+M$

$\Rightarrow ab + M = 1 + M$

$\Rightarrow ab - 1 \in M$

$\Rightarrow ab - 1 = m$ for some $m \in M$

$\Rightarrow 1 = ab - m \in I$ (using def. of ideal)

$\Rightarrow I = R$ (ideal containing unity, equals the ring)

Hence M is maximal ideal of R.

2. **If U, V are ideals of R and UV be the set of all elements that can be written as finite sum of elements of the form uv where $u \in U$, $v \in V$. Prove that UV is an ideal of R and UV $\subset$ U $\cap$ V.** **(September 2013, September 2012)**

Sol. U and V are ideals of a ring R

Let $UV = \{u_1v_1 + u_2v_2 + ... + u_nv_n : u_1, u_2.....u_n \in U, v_1, v_2.....v_n \in V\}$

Let $\alpha = u_1v_1 + u_2v_2 + ... + u_nv_n$

$\beta = u_1'v_1' + u_2'v_2' + ... + u_m'v_m'$ be any two elements of UV

$\alpha - \beta = u_1v_1 + u_2v_2 + ... + u_nv_n - u_1'v_1' - u_2'v_2' + ... + u_m'v_m'$

$= u_1v_1 + u_2v_2 + ... + u_nv_n + (-u_1')v_1' + (-u_2')v_2' + ... + (-u_m')v_m'$

$\Rightarrow \alpha - \beta$ is an element of UV because U and V are ideals and $u_1' \in U \Rightarrow (-u_1') \in U$ and so on.

Let $r \in R$ and $\alpha \in UV$ then

$r\alpha = r(u_1v_1 + ... + u_nv_n) = (ru_1)v_1 + ... + (ru_n)v_n$

$\Rightarrow r\alpha$ is an element of UV because U is an ideal and $r \in R$, $u_1 \in U \Rightarrow ru_1 \in U$ and so on.

Also $\alpha r = (u_1v_1 + ... + u_nv_n)r = u_1(v_1r) + ... + u_n(v_nr)$

$\Rightarrow \alpha r$ is an element of UV because V is an ideal and $r \in R$, $v_1 \in V \Rightarrow v_1r \in V$ and so on.

$\Rightarrow$ UV is an ideal of R.

Now to show $UV \subseteq U \cap V$

Let $\alpha = u_1v_1 + ... + u_nv_n$ be any element of UV where $u_1, ..., u_n \in U$ and $v_1, ..., v_n \in V$

Now $v_1 \in V \Rightarrow v_1 \in R$ and U is an ideal

$\therefore v_1 \in R, u_1 \in U \Rightarrow u_1v_1 \in U$

Similarly $u_1 \in U \Rightarrow u_1 \in R$ and V is an ideal

$\therefore u_1 \in R, v_1 \in V \Rightarrow u_1v_1 \in V$

$\Rightarrow u_1v_1 \in U, u_1v_1 \in V \Rightarrow u_1v_1 \in U \cap V$

Similarly $u_2v_2, ... u_nv_n \in U \cap V$

Since $U \cap V$ is an ideal of R

$\therefore u_1v_1, u_2v_2, ... u_nv_n \in U \cap V$

$\Rightarrow u_1v_1 + u_2v_2 + ... + u_nv_n \in U \cap V \Rightarrow \alpha \in U \cap V$

Thus $\alpha \in UV \Rightarrow \alpha \in U \cap V$

Hence $UV \subseteq U \cap V$

3. **If R is an Integral domain, then prove that $R[x]$ is also an integral domain.** **(September 2013, September 2012)**

Sol. Suppose R is an integral domain.

Let $f(x), g(x)$ be any two non zero members of $R[x]$ s.t.,

$f(x)g(x) = 0$

Let $f(x) = a_0 + a_1x + ... + a_mx^m$

and $g(x) = b_0 + b_1x + ... + b_nx^n$

Now both $f(x)$ and $g(x)$ cannot be constant polynomials as then $a_0 \neq 0$, $b_0 \neq 0$ (so $c_0 = a_0b_0 \neq 0$)

$\therefore f(x)g(x) \neq 0$

Since at least one of $f(x), g(x)$ is non constant polynomial, its degree is ≥ 1.

R being an integral domain

$\deg(f(x)g(x)) = \deg f(x) + \deg g(x) \geq 1$

Which is a contradiction as it implies then $c_k \neq 0$ for some $k > 0$

Whereas $f(x)g(x) = 0$.

Hence $f(x)g(x) = 0 \Rightarrow f(x) = 0$ or $g(x) = 0$

$\Rightarrow R[x]$ is an integral domain.

4. Find the field of quotients of the integral domain $\mathbb{Z}[\sqrt{2}]$.

Sol. Let F be the field of quotients of $\mathbb{Z}[\sqrt{2}]=\{a+b\sqrt{2}:a,b\in\mathbb{Z}\}$

then

$$F=\{xy^{-1}:x,y\in\mathbb{Z}[\sqrt{2}] \text{ and } y\neq 0\}$$

$$=\left\{\frac{a+\sqrt{2}b}{c+\sqrt{2}d}:a,b,c,d\in\mathbb{Z},\ c+\sqrt{2}d\neq 0\right\}$$

$$=\left\{\left(\frac{a+\sqrt{2}b}{c+\sqrt{2}d}\right)\left(\frac{c-\sqrt{2}d}{c-\sqrt{2}d}\right):a,b,c,d\in\mathbb{Z},\ c\neq 0,\ d\neq 0\right\}$$

$$=\left\{\left(\frac{ac-2bd}{c^2-2d^2}\right)+\sqrt{2}\left(\frac{bc-ad}{c^2-2d^2}\right):a,b,c,d\in\mathbb{Z},\ c^2-2d^2\neq 0\right\}$$

$\subseteq F_1$ where $F_1=\{p+\sqrt{2}q:p,q\in\mathbb{Q}\}$

$\therefore\ F\subseteq F_1$ (1)

Now $\forall p+\sqrt{2}q\in F_1$ where $p,q\in\mathbb{Q}$

Let $p=\frac{m_1}{n_1}$ and $q=\frac{m_2}{n_2}, m_1,m_2,n_1,n_2\in\mathbb{Z},\ n_1\neq 0,\ n_2\neq 0$

$$\therefore\ p+\sqrt{2}q=\frac{m_1}{n_1}+\sqrt{2}\frac{m_2}{n_2}=\frac{m_1n_2+\sqrt{2}m_2n_1}{n_1n_2+\sqrt{2}.0}\in F$$

$\Rightarrow F_1\subseteq F$ (2)

from (1) and (2), we get

$F=F_1$

Hence $F=\{p+\sqrt{2}q:p,q\in\mathbb{Q}\}$ is field of quotients of $\mathbb{Z}(\sqrt{2})$

5. If R is commutative ring with unity with characteristic 2 show that:

$(a+b)^2=a^2+b^2=(a-b)^2 \quad \forall a,b\in R$. **(April 2013, April 2010)**

Sol. R is commutative ring with characteristic 2

$\Rightarrow 2x=0\ \forall\ x\in$ R

and ab = ba $\forall$ a, b $\in$ R

Now $(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2$

$=a^2+2ab+b^2$ $(\because ab=ba)$

$=a^2+b^2$ $(\because 2ab=0)$

Also $(a-b)^2=(a-b)(a-b)$

$=a^2-ab-ba+b^2$

$=a^2-2ab+b^2$ $(\because ab=ba)$

$=a^2+b^2$ $(\because 2ab=0)$

Hence $(a+b)^2=a^2+b^2=(a-b)^2$

6. If R is an integral domain, then prove that $R[x]$; ring of polynomial over R is also an integral domain. **(April 2013)**

Sol. Suppose R is an integral domain.

Let $f(x),g(x)$ be any two non zero members of $R[x]$ s.t.,

$f(x)g(x)=0$

Let $f(x)=a_0+a_1x+...+a_mx^m$

and $g(x)=b_0+b_1x+...+b_nx^n$

Now both $f(x)$ and $g(x)$ cannot be constant polynomials as then $a_0\neq 0$, $b_0\neq 0$ (so $c_0=a_0b_0\neq 0$)

$\therefore\ f(x)g(x)\neq 0$

Since at least one of $f(x),g(x)$ is non constant polynomial, its degree is ≥ 1.

R being an integral domain

$\deg(f(x)g(x))=\deg f(x)+\deg g(x)\geq 1$

Which is a contradiction as it implies then $c_k\neq 0$ for some $k>0$

Whereas $f(x)g(x)=0$.

Hence $f(x)g(x)=0\Rightarrow f(x)=0$ or $g(x)=0$

$\Rightarrow R[x]$ is an integral domain.

7. Prove that for any two ideals I_1,I_2 of a ring R, I_1+I_2 is also an ideal of R containing both I_1 and I_2. **(April 2013)**

Sol. $I_1+I_2\neq\phi$ as $0=0+0\in I_1+I_2$

Again, let $x,y\in I_1+I_2$

$\Rightarrow x=a_1+b_1$

$y=a_2+b_2$ for some $a_1,a_2\in I_1;b_1,b_2\in I_2$

Since $x-y=(a_1+b_1)-(a_2+b_2)$

$=(a_1-a_2)+(b_1-b_2)$

We find $x-y\in I_1+I_2$

Let $x=a+b\in I_1+I_2$, $r\in R$ be any element then

$xr=(a+b)r=ar+br\in I_1+I_2$ as I_1 *and* I_2 are ideals

also $rx=r(a+b)=ra+rb\in I_1+I_2$

Thus I_1+I_2 is an ideal of R

Again for any $a\in I_1$, since

$a=a+0\in I_1+I_2$ and

for any $b\in I_2$

$b=0+b\in I_1+I_2$

We find $I_1\subseteq I_1+I_2$

and $I_2\subseteq I_1+I_2$

8. A commutative ring R with unity is simple if and only if R is a field. **(September 2012)**

Sol. R is commutative simple ring with unity.

To show R is field we need to show $<R-\{0\},.>$ is a group

Let $x(\neq 0)\in R$

Consider $xR=\{xr\mid r\in R\}$

Clearly $x=x.1\in xR$ so xR is non-empty.

Further $xy-xz=x(y-z)\in xR$ and

for any $s\in R$, $(xr)s=x(rs)\in xR$

$\Rightarrow xR$ is a right ideal of R

Since $(x\neq 0)\in xR$, $xR=R$

As $1\in R$, there exists $y\in R$ such that xy = 1. Thus $<R-\{0\},.>$ is semi group with unity in which every element is right invertible. Hence $<R-\{0\},.>$ is a group. Consequently R is division ring.

$\Rightarrow$ Commutative division ring R is field.

Conversely,

Let R be a field.

$\Rightarrow$ R is commutative division ring

$\Rightarrow$ R has unity

We will show division ring is simple ring.

Let A be any ideal of R s.t. $A\neq\{0\}$

then $\exists$ atleast one $a\in A$ s.t. $a\neq 0$

R being a division ring, $a^{-1}\in R$

and $aa^{-1}=1$

Since $a\in A$, $a^{-1}\in R$, $aa^{-1}\in A$

$\Rightarrow 1\in A$

Also $A\subseteq R$ we need to show $R\subseteq A$

Let $r\in R$ be any element

Since $1\in A$ and A is an ideal

$r=1.r\in A$

$\Rightarrow R\subseteq A \Rightarrow R=A$

i.e. only ideals that R can have are R and $\{0\}$

$\Rightarrow$ R is commutative simple ring with unity.

9. Show that for every prime p, the ring $\frac{Z}{pZ}$ with the usual modulo operations, is a field. **(April 2012)**

Sol. Let p be prime

To show $\frac{Z}{pZ}=\{0,1,2,........,p-1\}$ is a field

We need to show $\frac{Z}{pZ}$ is an integral domain

$\because$ finite integral domain will be a field.

Let $a\otimes b=0$ $a,b\in\frac{Z}{pZ}$

$\Rightarrow$ ab is multiple of p

$\Rightarrow$ p|ab

$\Rightarrow$ p|a or p|b (p being prime)

$\Rightarrow$ a = 0 or b = 0 $\quad(\because a,b\in\frac{Z}{pZ}$ and a, b < p)

$\Rightarrow\frac{Z}{pZ}$ is an integral domain

But $\frac{Z}{pZ}$ is finite integral domain

$\Rightarrow\frac{Z}{pZ}$ is field.

10. Show that an ideal P in ring R is prime ideal if and only if $\frac{R}{P}$ is an integral domain. **(April 2012, 2011, 2010)**

Sol. Let P be a prime ideal of R

Let $(a+P)(b+P)=0+P$

Then $ab+P=P$

$\Rightarrow ab \in P$

$\Rightarrow a \in P$ or $b \in P$

$\Rightarrow a+P=P$ or $b+P=P$

thus $\frac{R}{P}$ is an integral domain.

Conversely, let $\frac{R}{P}$ be an integral domain.

Let $ab \in P$ then $ab+P=P$

$\Rightarrow (a+P)(b+P)=P$

$\Rightarrow a+P=P$ or $b+P=P$ ($\frac{R}{P}$ is integral domain)

$\Rightarrow a \in P$ or $b \in P$

Hence the result.

11. What are the units of the polynomial ring $Z_7[x]$? **(April 2012)**

Sol. First we will show units of R[x] are units of R.

Clearly a unit of R is a unit of R[x] as R $\subseteq$ R[x] and they have same unity.

Conversely let $f \in R[x]$ be a unit in $R[x]$ then there exists $g \in R[x]$ such that fg = 1.

This implies deg f + deg g = 0 $\Rightarrow$ def f = deg g = 0 $\Rightarrow$ f and g are constants. Thus f is unit of R

Now $Z_7 = \{0,1,2,3,4,5,6\}$

Units of Z_7 are 1, 2, 3, 4, 5, 6

$\because$ 1.1 = 1

3.5 = 1

2.4 = 1

6.6 = 1

Hence units of $Z_7[x]$ are 1, 2, 3, 4, 5, 6

12. For any two ideals A and B of a ring R, prove that A + B = <AUB>. **(September 2011)**

Sol. $A+B \neq \phi$ as $0 = 0+0 \in A+B$

Again, let $x, y \in A+B$

$\Rightarrow x = a_1 + b_1$

$y = a_2 + b_2$ for some $a_1, a_2 \in A$; $b_1, b_2 \in B$

Since $x - y = (a_1+b_1)-(a_2+b_2)$

$=(a_1-a_2)+(b_1-b_2)$

We find $x-y \in A+B$

Let $x=a+b \in A+B, r \in R$ be any elements then

$xr=(a+b)r=ar+br \in A+B$ as A, B are ideals

$rx=r(a+b)=ra+rb \in A+B$

Thus A + B is an ideal of R.

Again for any $a \in A$, since $a=a+0 \in A+B$ and for any $b \in B$, since $b=0+b \in A+B$

We find $A \subseteq A+B$

$B \subseteq A+B$.

We have already proved that A + B is an ideal of R, containing A and B, thus A + B is an ideal containing $A \cup B$.

To prove, A + B is the smallest ideal containing $A \cup B$.

Let I be any ideal of R, s.t., $A \cup B \subseteq I$

Let $x \in A+B$ be any element

Then $x=a+b$ for some $a \in A$, $b \in B$

Since $a \in A \subseteq A \cup B \subseteq I$

$b \in B \subseteq A \cup B \subseteq I$

We find $a+b \in I$ as I is an ideal

$\Rightarrow x \in I$ or that $A+B \subseteq I$

which proves the theorem.

Thus A + B is the smallest ideal of R, containing A and B.

i.e. $A + B = \langle A \cup B \rangle$

13. Let T be a ring of polynomials over a ring R. Prove that $R' = \{(a,0,0,.....) : a \in R\}$ is a subring of T which is isomorphic to R. Further if R has unity 1, then what is the unity of T? **(September 2011)**

Sol. To prove R' is a subring T.

We need to show that

$x-y\in R'$ and $xy\in R'$ where $x,y\in R'$

Let $x=(a,0,0,..........)$ and $y=(b,0,0,..........)$

Now $x-y=(a,0,0,..........)-(b,0,0,..........)$

$=(a-b,0,0,.......)\in R'$ $[\because a-b\in R]$

Also $xy=(a,0,0,..........)(b,0,0,..........)$

$=(ab,0,0,.......)\in R'$ $[\because ab\in R]$

Hence R' is a subring of T.

Let $f:R\to R'$ be defined by

$f(a)=(a,0,0,......)$ $\forall a\in R$

Thus for all $a,b\in R$

$f(a+b)=(a+b,0,0,......)=(a,0,0,......)+(b,0,0,......)$

$=f(a)+f(b)$

$f(ab)=(ab,0,0,......)=(a,0,0,......)(b,0,0,......)$

$=f(a)f(b)$

Also f is 1 – 1

Since $f(a)=f(b)$

$\Rightarrow(a,0,0,......)=(b,0,0,......)$

$\Rightarrow$ a = b

Hence $R\cong R'$

If R has unity 1 then

$f(1)=(1,0,0,......)\in R'\subseteq T$ and for

all $(a_1,a_2,a_3,......)\in T$

$(a_1,a_2,a_3,.........)(1,0,0,.........)$

$=(a_1,a_2,a_3,.........)=(1,0,0,.........)(a_1,a_2,a_3,.........)$

Hence $(1,0,0,.........)$ is unity of T.

14. Let R be a ring with unity such that $(xy)^2=x^2y^2$ for all $x,y\in R$. Prove that R is commutative. **(April 2011)**

Sol. Let x, y, ∈ R be any elements

Then y + 1 ∈ R as 1 ∈ R

By given condition

$(x(y+1))^2=x^2(y+1)^2$

$\Rightarrow (xy+x)^2=x^2(y+1)^2$

$\Rightarrow (xy+x)(xy+x)=x^2(y+1)(y+1)$

$\Rightarrow (xy)^2+x^2+xyx+xxy=x^2(y^2+1+2y)$

$\Rightarrow x^2y^2+x^2+xyx+xxy=x^2y^2+x^2+2x^2y$

$\Rightarrow xyx=x^2y$ **(1)**

Since (1) holds for all x, y in R, it holds for x + 1, y also. Thus replacing x by x + 1, we get

$(x+1)y(x+1)=(x+1)^2y$

$\Rightarrow (xy+y)(x+1)=(x^2+1+2x)y$

$\Rightarrow xyx+xy+yx+y=x^2y+y+2xy$

$\Rightarrow yx=xy$ using (1)

Hence R is commutative.

15. Let A and B be any two ideals of a ring R. Prove that A ∪ B is an ideal of R if and only if either A ⊆ B or B ⊆ A. **(April 2011)**

Sol. If $A\subseteq B$ then $A\cup B=B$ is an ideal of R.

Also $B\subseteq A$ then $A\cup B=A$ is an ideal of R.

Conversely, Let $A\cup B$ be an ideal of R. Suppose neither $A\not\subseteq B$ nor $B\not\subseteq A$

then there exists elements x ∈ A – B and y ∈ B – A

Now x ∈ A and y ∈ B

$\Rightarrow$ x, y, ∈ $A\cup B$

Also $A\cup B$ is an ideal which implies x – y ∈ $A\cup B$ so either x – y ∈ A or x – y ∈ B

if x – y ∈ A then y = x – (x – y) ∈ A

which contradicts y ∉ A

Similarly if $x-y\in B$ then $x=(x-y)+y\in B$

which contradicts $x\notin B$

Hence our supposition is wrong

∴ either $A\subseteq B$ or $B\subseteq A$

16. Let F be a field. Let $f,g\in F[x], g\neq 0$. Prove that there exist unique polynomials h, $r\in F[x]$ such that $f=hg+r$ where either r = 0 or degree r < degree g. **(April 2011)**

Sol. Suppose

$f(x)=a_0+a_1x+a_2x^2+....+a_mx^m, a_m\neq 0$

and $g(x)=b_0+b_1x+b_2x^2+....+b_nx^n, b_n\neq 0$

If degree m of f(x) is smaller than the degree n of g(x) or if $f(x)=0$. $g(x)+f(x)$. So in this case $q(x)=0$, $r(x)=f(x)$ and we have either $f(x)=0$ or $\deg f(x)<\deg g(x)$.

Now let us assume that $m\geq n$. In this case we shall prove the theorem by induction on m i.e., degree of $f(x)$.

If $m=0$, then $m\geq n\Rightarrow n=0$. Therefore $f(x)$ and $g(x)$ are both non-zero constant polynomials, $f(x)=a_0$, $a_0\neq 0$ and $g(x)=b_0$, $b_0\neq 0$. We have in this case

$f(x)=a_0=(a_0b_0^{-1})b_0+0=(a_0b_0^{-1})g(x)+0$.

Thus the theorem is true when m = 0 or when the degree of *f(x)* is less than 1.

We shall now assume that the theorem is true when *f(x)* is a polynomial of degree less than m and then we shall show that it is also true if $f(x)$ is of degree m and then the proof will be complete by induction.

Let $f_1(x)=f(x)-(a_mb_n^{-1})x^{m-n}g(x)$ (1)

Obviously $\deg f_1(x)<m$. Therefore by our assumed hypothesis, there exist polynomials $s(x)$ and $r(x)$ such that

$f_1(x)=s(x)g(x)+r(x)$,

Where $r(x)=0$ or $\deg r(x)<\deg g(x)$.

Now putting the value of $f_1(x)$ in (1), we get

$s(x)g(x)+r(x)=f(x)-(a_mb_n^{-1})x^{m-n}g(x)$

or $f(x)=\left[(a_mb_n^{-1})x^{m-n}+s(x)\right]g(x)+r(x)$.

If we write $q(x)$ in place of $(a_mb_n^{-1})x^{m-n}+s(x)$, we get

$f(x)=q(x)g(x)+r(x)$

where $r(x)=0$ or $\deg r(x)<\deg g(x)$.

This proves the existence of polynomials $q(x)$ and $r(x)$.

Now to show that $q(x)$ and $r(x)$ are unique. Let us assume that q(x) and r(x) are not unique i.e. there exists $q_1(x)$, $q_2(x)$ and $r_1(x)$, $r_2(x)$ such that

$f(x)=q_1(x)g(x)+r_1(x)$

where deg. $r_1(x)=0$ or deg $r_1(x)<\deg g(x)$

and $f(x)=q_2(x)g(x)+r_2(x)$

where deg. $r_2(x)=0$ or deg $r_2(x)<\deg g(x)$

$\Rightarrow 0=(q_1(x)-q_2(x))g(x)+(r_1(x)-r_2(x))$

or $(q_1(x)-q_2(x))g(x)=(r_2(x)-r_1(x))$ (2)

If $[q_1(x)-q_2(x)]\neq 0$, then $[q_1(x)-q_2(x)]g(x)$ cannot be equal to the zero polynomial because $g(x)\neq 0$ and $F[x]$ is without zero divisors. Also then the degree of $[q_1(x)-q_2(x)]g(x)$ is at least n, the degree of $g(x)$. But $r_2(x)-r_1(x)$ is either equal to the zero polynomial or else its degree is less than n because the degrees of $r_2(x)$ and $r_1(x)$ are both less than n. Hence the equality (2) among two polynomials holds only if

$q_1(x)-q_2(x)=0$ and $r_2(x)-r_1(x)=0$

i.e., only if $q_1(x)=q_2(x)$ and $r_2(x)=r_1(x)$.

$\therefore$ the polynomials $q(x)$ and $r(x)$ are unique.

17. For any two ideals A and B of a ring R prove that $AB=\{\sum a_ib_i | a_i\in A,\ b_i\in B\}$ is an ideal of R. (September 2010)

Sol. $AB\neq\varphi$ as $0=0.0\in AB$

Let $x,y\in AB$ be any two members.

Then $x=a_1b_1+a_2b_2+....+a_nb_n$

$y=a_1'b_1'+a_2'b_2'+....+a_m'b_m'$

For some $a_i,a_j'\in A, b_i,b_j'\in B$

Now, $x-y=(a_1b_1+a_2b_2+....+a_nb_n)-(a_1'b_1'+a_2'b_2'+....+a_m'b_m')$

which clearly belongs to AB, as the R.H.S. can be written as

$x_1y_1+x_2y_2+......+x_ky_k$ $(k=n+m)$

where $x_i\in A$, $y_i\in B$.

Again, for any $x=a_1b_1+....+a_nb_n\in AB$ and $r\in R$,

$rx=r(a_1b_1+.......a_nb_n)$

$=(ra_1)b_1+(ra_2)b_2+.....+(ra_n)b_n\in AB$

because $ra_i\in A$ as $a_i\in A$, $r\in R$ and A is an ideal.

Similarly $xr\in AB$

showing thereby that AB is an ideal of R.

18. IF N is an ideal of R then prove that there is 1 – 1 correspondence between ideals of R containing N and ideals of R/N. (September 2010)

Sol. Let $f: R \to R/N$ be the natural homomorphism defined by $f(r) = r + N$. Now, if A be any ideal of R then as $f: R \to R/N$ is onto homomorphism, $f(A)$ is an ideal of R/N.

Again, $f(A) = \{f(a) \mid a \in A\}$

$= \{a + N \mid a \in A\}$

$= \frac{A}{N}$.

Let know K be the set of all ideals of R, which contain N and K' be the set of all ideals of $\frac{R}{N}$.

Define $\varphi: K \to K'$ s.t.,

$\varphi(A) = f(A) = \frac{A}{N}$

φ is clearly well-defined.

Again $\varphi(A) = \varphi(B)$

$\Rightarrow f(A) = f(B)$

$\Rightarrow \frac{A}{N} = \frac{B}{N}$

If $a \in A$ be any element then $a + N \in \frac{A}{N} \Rightarrow a + N \in \frac{B}{N}$

$\Rightarrow a + N = b + N$ for some $b \in B$

$\Rightarrow a - b \in N \subseteq B$

$\Rightarrow a - b = b'$ for some $b' \in B$

and thus $\Rightarrow a = b + b' \in B$

i.e., $A \subseteq B$.

Similarly, $B \subseteq A$ and thus A = B

Showing that φ is one-one.

To show that φ is onto, let $X \in K'$ be any member then X is an ideal of $\frac{R}{N}$.

Define $A = \{x \in R \mid f(x) \in X\}$.

We show A is the required pre-image of X under φ.

It is easy to check that A is an ideal of R.

Again, $n \in N = \text{Ker} f$

$\Rightarrow f(n) = N$ = zero of $\frac{R}{N}$

$0 + N \in X$ [as ideal contains zero]

$\therefore f(n) \in X \Rightarrow n \in A$

or that $N \subseteq A$.

Thus A is a member of K.

Definition of A then confirms that it is the required pre-image. Hence φ is onto.

19. If R is a commutative ring, then prove that R[x] is commutative. Further prove that if R has no proper zero divisors then R[x] also has no proper zero divisors. (September 2010)

Sol. (i) If R[x] is commutative then any subring of R[x] is commutative and as R is isomprphic to a subring of R[x], R will be commutative.

Conversely, if R is commutative

and $f(x) = a_0 + a_1x + a_2x^2 + ... + a_mx^m$

$g(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^n$

be two members of R[x], then by definition of product

$f(x)g(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + ...$

$= b_0a_0 + (b_1a_0 + b_0a_1)x + ...$

$= g(x)f(x)$.

For second part, it is given R has no proper zero divisors which means R is integral domain so we will show that if R is integral domain then R[x] is integral domain

Suppose R is an integral domain.

Let $f(x), g(x)$ be any two non zero members of R[x] s.t.,

$f(x)g(x) = 0$

where $f(x) = a_0 + a_1x + ... + a_mx^m$

$g(x) = b_0 + b_1x + ... + b_nx^n$

Now both $f(x)$ and $g(x)$ cannot be constant polynomials as then $a_0 \neq 0$, $b_0 \neq 0$ (so $c_0 = a_0b_0 \neq 0$)

$\therefore f(x)g(x) \neq 0$

Since at least one of $f(x), g(x)$ is non constant polynomial, its degree is ≥ 1.

R being an integral domain

$\deg(f(x)g(x)) = \deg f(x) + \deg g(x) \geq 1$

which is a contradiction as it implies then $c_k \neq 0$ for some $k > 0$

whereas $f(x)g(x) = 0$

Hence $f(x)g(x)=0 \Rightarrow f(x)=0$ or $g(x)=0$

$\Rightarrow$ R[x] is an integral domain.

20. Define simple ring. Prove that a division ring is a simple ring. **(April 2010)**

Sol. Definition: A ring $R \neq \{0\}$ is called a simple ring if R has no ideals except R and $\{0\}$.

Now, let R be a division ring.

Let A be any ideal of R s.t., $A \neq \{0\}$ then $\exists$ at least one $a \in A$ s.t., $a \neq 0$.

R being a division ring, $a^{-1} \in R$ and $aa^{-1}=1$.

Since $a \in A$, $a^{-1} \in R \Rightarrow aa^{-1} \in A$ [$\because$ A is ideal of R]

$\Rightarrow 1 \in A$

Since $A \subseteq R$ always, all we need show is that $R \subseteq A$. Let $r \in R$ be any element.

Since $1 \in A$ and A is an ideal

$r = 1 . r \in A$

$\Rightarrow R \subseteq A$ or that A = R.

i.e., only ideas that R can have are R and $\{0\}$ or that R is a simple ring.

21. For any positive integer n, prove that ring $Z/_n$ of all integers modulo n is an integral domain if and only if n is a prime integer. **(September 2009)**

Sol. Let $Z_n = \{1, 2, ..., (n-1)\}$ modulo n is an Integral domain.

Suppose n is not a prime then $\exists$ a, b such that n = ab

$1 < a, b < n$

$\Rightarrow a \otimes b = 0$ where a, b are non zero

$\Rightarrow Z_n$ has zero divisors

$\Rightarrow Z_n$ is not integral domain, a contradiction

Conversely,

Let n be a prime and to show Z_n is an integral domain

Let $a \otimes b = 0$ $\qquad a, b \in Z_n$

Thus ab is a multiple of n

$\Rightarrow$ n|ab

$\Rightarrow$ n|a or n|b (n being prime)

$\Rightarrow a = 0$ or $b = 0$ $(a, b \in Z_n \Rightarrow a, b < n)$

$\Rightarrow Z_n$ is an integral domain.

22. If R is a ring in which $x^2 = x\ \forall x \in R$ prove that R is a commutative ring of characteristic two. **(April 2009)**

Sol. Let R be ring such that $x^2 = x\ \forall x \in R$

$x \in R \Rightarrow x + x \in R$

$\Rightarrow (x+x)^2 = x + x \qquad (\because x^2 = x\ \forall x \in R)$

$\Rightarrow (x+x)(x+x) = x + x$

$\Rightarrow x^2 + x^2 + x^2 + x^2 = x + x$

$\Rightarrow x + x + x + x = x + x$

$\Rightarrow x + x = 0$

$\Rightarrow 2x = 0 \qquad \forall x \in R$

$\Rightarrow$ R is ring with characteristic 2

Let $x + y = 0$, we have just proved $x + x = 0$

$\Rightarrow x + x = x + y \Rightarrow x = y \qquad (1)$

Also $(x+y)^2 = x + y \Rightarrow (x+y)(x+y) = x + y$

$\Rightarrow x^2 + xy + yx + y^2 = x + y$

$\Rightarrow x + xy + yx + y = x + y \Rightarrow xy + yx = 0$

And because of (1) $xy = yx$.

Hence R is commutative.

23. Let A and B be two ideals of a ring R, then prove that $\dfrac{A}{(A \cap B)} \cong \dfrac{(A+B)}{B}$. **(September 2009, April 2009)**

Sol. Define a mapping $f : B \to \dfrac{A+B}{A}$ s.t.,

$f(b) = b + A$ for all $b \in B$

Then f is well defined homomorphism.

Again if $x + A \in \dfrac{A+B}{A}$ be any element then

$x \in A + B \Rightarrow x = a + b,\ a \in A,\ b \in B$

So, $x + A = (a+b) + A = (b+a) + A = b + (a + A) = b + A$

thus $x + A = b + A = f(b)$

i.e., b is the pre-image of $x + A$ under f or that f is onto.

By fundamental theorem then $\dfrac{A+B}{A} \cong \dfrac{B}{\text{Ker} f}$

Now $x \in \text{Ker} f \Leftrightarrow f(x) = A$

$\Leftrightarrow x + A = A \Leftrightarrow x \in A$

$\Leftrightarrow x \in A \cap B \qquad (x \in \text{Ker} f \subseteq B)$

Hence $\quad \text{Ker}\ f = A \cap B$

and thus $\dfrac{A+B}{A} \cong \dfrac{B}{A \cap B}$

3

VECTOR SPACES

1. **Let W_1 and W_2 be two subspaces of V_F, then the sum $W_1+W_2=\{w_1+w_2 \mid w_1\in W_1, w_2\in W_2\}$ is a subspace of V containing $W_1 \cup W_2$.**

(September 2013, September 2012)

Sol. Let $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ so that

$x_1+x_2 \in W_1+W_2$ and $y_1+y_2 \in W_1+W_2$

Now for $\alpha,\beta \in F$ and $x_1, y_1 \in W_1$

$\Rightarrow \alpha x_1+\beta y_1 \in W_1$ [$\because$ W_1 is is subspace of V(F)]

Also for $\alpha,\beta \in F$ and $x_2, y_2 \in W$

$\Rightarrow \alpha x_2+\beta y_2 \in W_2$ [$\because$ W_2 is is subspace of V(F)]

Since $(\alpha x_1+\beta y_1)\in W_1$ and $(\alpha x_2+\beta y_2)\in W_2$

$\therefore$ $(\alpha x_1+\beta y_1)+(\alpha x_2+\beta y_2)\in W_1+W_2$

Now for $\alpha,\beta \in F$

$\alpha(x_1+x_2)+\beta(y_1+y_2)=ax_1+ax_2+\beta y_1+\beta y_2$

$=(\alpha x_1+\beta y_1)+(\alpha x_2+\beta y_2)\in W_1+W_2$

Hence W_1+W_2 is a subspace of V(F)

Now to prove that $W_1+W_2=\{W_1\cup W_2\}$

Let $x_1\in W_1$ $\Rightarrow x_1+0\in W_1+W_2$ [$\because 0\in W_2$]

$\Rightarrow x_1\in W_1+W_2$

$\therefore$ $W_1\subset W_1+W_2$ (1)

Also $x_2\in W_2$ $\Rightarrow 0+x_2\in W_1+W_2$ [$\because 0\in W_1$]

$\Rightarrow x_2\in W_1+W_2$

$\therefore$ $W_2\subset W_1+W_2$ (2)

From (1) and (2),

$W_1\cup W_2\subset W_1+W_2$

Since $\{W_1\cup W_2\}$ is the smallest subspace

$\therefore$ $\{W_1\cup W_2\}\subset W_1+W_2$ (3)

Now $\because x_1\in W_1, x_2\in W_2$

$\therefore$ $x_1, x_2\in W_1\cup W_2$

$\Rightarrow x_1+x_2$ lie in subspace $\{W_1\cup W_2\}$

i.e. $W_1+W_2\subset\{W_1\cup W_2\}$ (4)

From (3) and (4), we have

$W_1+W_2=\{W_1\cup W_2\}$

Hence the proof.

2. **Extend the set $\{(-1,2,5)\}$ to two different basis of $R^3(R)$.** **(September 2013)**

Sol. Let $B=\{(1,0,0),(0,1,0),(0,0,1)\}=\{e_1,e_2,e_3\}$ be a basis of $R^3(R)$.

Given set $\{(-1,2,5)\}$ L.I. over R, being non-zero vector

$\therefore$ the vector v, e_1, e_2, e_3 span R^3 where $v=(-1,2,5)$

Since dim $R^3=3$, so any basis of R^3 contains exactly three L.I. vectors

To find these vectors,

Let $A=\begin{bmatrix}-1 & 2 & 5\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ Operate $R_2\to R_2+R_1$

$\sim\begin{bmatrix}-1 & 2 & 5\\ 0 & 2 & 5\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ Operate $R_3\to 2R_3$

$\sim\begin{bmatrix}-1 & 2 & 5\\ 0 & 2 & 5\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$ Operate $R_3\to R_3-R_2$

$$\sim\begin{bmatrix}-1 & 2 & 5\\ 0 & 2 & 5\\ 0 & 0 & -5\\ 0 & 0 & 1\end{bmatrix} \qquad \text{Operate } R_4 \to R_4 + \frac{1}{5}R_3$$

$$\sim\begin{bmatrix}-1 & 2 & 5\\ 0 & 2 & 5\\ 0 & 0 & -5\\ 0 & 0 & 0\end{bmatrix}$$

which is echelon form of A, having three non zero rows, so they form a basis of R^3

$\therefore$ the set $B=\{(-1,2,5),(0,2,5),(0,0,-5)\}$ is a basis of $R^3(R)$.

Also the set $B_2=\{(-1,2,5),(0,1,0),(0,0,1)\}$ is a basis of $R^3(R)$

[$\because$ these form rows of a echelon matrix]

Thus, we have two different basis, which are extension of $\{(-1,2,5)\}$

3. **Prove that union of two subspaces U_1, U_2 of $V(F)$ is a subspace of V(F) if either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.** **(April 2013, April 2012, September 2011, April 2009)**

Sol. Given: $U_1 \cup U_2$ is a sub-pace of V(F)

To prove: Either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

Let us suppose that neither $U_1 \subseteq U_2$ nor $U_2 \subseteq U_1$

Now U_1 is not a subset of U_2 (1)

$\Rightarrow \exists x \in U_1$ such that $x \notin U_2$

Also U_2 is not a subset of U_1

$\Rightarrow \exists y \in U_2$ such that $y \notin U_1$ (2)

Further $x \in U_1 \Rightarrow x \in U_1 \cup U_2$

And $y \in U_2 \Rightarrow y \in U_1 \cup U_2$ $\left[\because \begin{matrix} U_1 \subset U_1 \cup U_2 \\ U_2 \subset U_1 \cup U_2 \end{matrix}\right]$

But $U_1 \cup U_2$ is a subpace of V(F) [Given]

$\therefore x+y \in U_1 \cup U_2$

$\Rightarrow x+y \in U_1$ or $x+y \in U_2$ (3)

If $x+y \in U_1$ and $x \in U_1$

Then $(x+y)-x \in U_1$ [$\because U$ is a subsapce]

$\Rightarrow y \in U_1$ which contradicts (2)

$\therefore x+y \notin U_1$ (4)

Also if $x+y \in U_2$ and $y \in U_2$

Then $(x+y)-y \in U_2$ [$\because U_2$ is a subspace]

$\Rightarrow x \in U_2$ which contradicts (1)

$\therefore x+y \notin U_2$ (5)

Now (4) and (5) contradict (3)

$\therefore$ Our supposition is wrong

Hence either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

Conservely. Given : Either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

To prove : $U_1 \cup U_2$ is a subspace of V(F)

Since $U_1 \subseteq U_2$

$\therefore U_1 \cup U_2 = U_2$

But U_2 is a subspace of V(F)

$\therefore U_1 \cup U$ is a space of V(F)

Also $\because U_2 \subseteq U_1$

$\therefore U_1 \cup U_2 = U_1$

But U_1 is subspace of V(F)

$\therefore U_1 \cup U_2$ is a subspace of V(F)

Thus either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$

$\Rightarrow U_1 \cup U_2$ is a subspace of V(F)

Hence the proof.

4. **For any two subset S and T of V(F). Show that:** **(April 2013)**
(i) $S \subset T \Rightarrow L(S) \subset L(T)$
(ii) L(S) = S iff S is subspace of V(F).

Sol. (i) Let $x \in L(S)$

$\therefore x = \sum_{i=1}^{n} a_i x_i$ where a_i's $\in$ F and x_i's $\in$ S

$\Rightarrow x \in L(T)$ $[\because S \subset T, \therefore \{x_1, x_2, ..., x_n\}$ is a finite subset of T$]$

Thus x $\in$ L(S) $\Rightarrow$ x$\in$L(T)

$\therefore L(S) \subset L(T)$

(ii) Let S be a subspace of V(F)

To prove that L(S) = S

Let x$\in$L(S)

Then $x=\sum_{i=1}^{n} a_i x_i$ where $a_i's \in F$ and $x_i's \in S$, $1 \le i \le n$

Since S is a subspace of V(F)

$\therefore$ S is closed w.r.t. vector addition and scalar multiplication.

$\therefore x \in S$

Thus $x \in L(S)$ $\Rightarrow x \in S$

$\therefore L(S) \subseteq S$

Also $S \subseteq L(S)$

$\therefore L(S) = S$

Conversely. Let L(S) = S

To prove that S is a subspace of V(F).

We know that L(S) is a subspace of V(F)

Let $x, y \in L(S)$

Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$

And $y = \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_m y_m$

Where $\alpha_i \in F, x_i \in S$ for $1 \le i \le n$

And $\beta_j \in F, y_j \in S$ for $1 \le j \le m$

First, to show that L(S) is a subspace of V(F)

Let $\alpha, \beta \in F$ and $x, y \in L(S)$

Now $\alpha x + \beta y = \alpha(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) + \beta(\beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_m y_m)$

$\Rightarrow \alpha x + \beta y = (\alpha\alpha_1)x_1 + (\alpha\alpha_2)x_2 \ldots + (\alpha\alpha_n)x_n + (\beta\beta_1)y_1 + (\beta\beta_2)y_2 + \ldots + (\beta\beta_m)y_m$

Thus $\alpha x + \beta y$ has been expressed as the linear combination of a finite set $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_m$ of the elements of S.

$\therefore \alpha x + \beta y \in L(S)$

Thus for $x, y \in L(S)$ and $\alpha, \beta \in F$, we have,

$\alpha x + \beta y \in L(S)$

Hence L(S) is a subspace of V(F).

$\therefore$ S is subspace of V $[\because L(S) = S]$

Hence the proof.

$\therefore$ S is a subspace of V $[\because L(S) = S]$

Hence the proof.

5. Show that $\exists$ a basis for each finite dimensional vector space.

(April 2013, September 2009, April 2009)

Sol. Let V(F) be a finite dimensional vector space.

$\therefore \exists$ a finite subset $S = \{v_1, v_2, \ldots, v_n\}$ of V such that

L(S) = V

Without any loss of generality we may assume that all the vectors in S are non-zero.

Now since $S \subset V$, so either S is L.I. or L.D.

If S is L.I., then S will be a basis of V(F)

$\therefore$ the theorem is proved. $[\because L(S) = V]$

If S is L.D., then some vector $v_k \in S$, $2 \le k \le n$, can be expressed as the linear combination of its preceding vectors

i.e. $v_k = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1}$ for $\alpha_i's \in F$

Consider the set

$S_1 = \{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$ of $(n-1)$ vectors.

Clearly $S_1 \subset S$

$\Rightarrow L(S_1) \subset L(S)$

$\Rightarrow L(S_1) \subset V$ $[\because L(S) = V]$ (1)

Since L(S) = V

$\therefore v \in V$

$\Rightarrow v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_{k-1} v_{k-1} + \beta_k v_k + \beta_{k+1} v_{k+1} + \ldots + \beta_n v_n$ $\beta_i's \in F$

$\Rightarrow v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_{k-1} v_{k-1} + \beta_k(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1}) + \beta_{k+1} v_{k+1} + \ldots + \beta_n v_n = (\beta_1 + \beta_k \alpha_1) v_1 + (\beta_2 + \beta_k \alpha_2) v_2 + \ldots + (\beta_{k-1} + \beta_k \alpha_{k-1}) v_{k-1} + \beta_{k+1} v_{k+1} + \ldots + \beta_n v_n$

$\Rightarrow$ v is linear combination of elements of S_1

$\Rightarrow v \in L(S_1)$

Now $v \in V$ $\Rightarrow v \in L(S_1)$

Thus $V \subset L(S_1)$ (2)

From (1) and (2)

$L(S_1) = V$

If S_1 is L.I. then S_1 is a basis of set V and theorem is proved.

If S_1 is L.D. then some vector $v_i \in S_1, i > k$, can be expressed as the linear combination of its preceding vectors i.e.

$v_i = \gamma_1 v_1 + \gamma_2 v_2 + \ldots + \gamma_{k-1} v_{k-1} + \gamma_{k+1} v_{k+1} + \ldots + \gamma_{i-1} v_{i-1}$

Consider a set $S_2 = \{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$

As we have proved above that $L(S_1) = V$, on similar lines we can show that $L(S_2) = V$.

If S_2 is L.I. then S_2 is a basis of V and the theorem is proved

If S_2 is L.D. then we repeat the above procedure till we get a set which is L.I. and which span V, thus giving a basis of V. At the most by repeating the procedure we can get a basis set of V which contains only single non-zero vector. [$\because$ the set of single non-zero vector is L.I.]

Hence there exists a basis for each finite dimensional vector space.

6. **Show that set $\{(2,1,4)(1,-1,2)(3,1,-2)\}$ form basis of R^3.** **(April 2013)**

Sol. As $\dim R^3 = 3$. Thus to show B is a basis of R^3, it is sufficient to show that B is L.I. set

Let $a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = 0$

$\therefore\ 2a_1 + a_2 + 3a_3 = 0$

$a_1 - a_2 + a_3 = 0$

$4a_1 + 2a_2 - 2a_3 = 0$

Eqns. can be written (in matrix form) as

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = 0$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$

Now det. A= det $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$

$$= 2\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix}$$

$= 2(2-2) - 1(-2-4) + 3(2+4)$

$= 0 + 6 + 18 = 24 \neq 0$

$\therefore$ B is L.I. set

$\Rightarrow$ B is basis for R^3.

7. **In the vector space R^3, let $x = (1,2,1), y = (3,1,5), z = (3,-4,7)$. Prove that subspaces spanned by $S = \{x, y\}, T = \{x, y, z\}$ are same.**

(September 2009, April 2009)

Sol. We know L(T) = The linear span of T is a set of vectors which is the linear combination of the vectors of T

$\therefore$ L(T) = $\{ax + by + cz \mid a, b, c \in R\}$ (1)

We shall express z = (3, –4, 7) as a linear combination of

x = (1, 2, 1) and y = (3, 1, 5)

Let $z = \alpha_1 x + \alpha_2 y$ for some scalars

$\Rightarrow (3,-4,7) = \alpha_1(1,2,1) + \alpha_2(3,1,5)$

$\Rightarrow (3,-4,7) = (\alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + 5\alpha_2)$

$\therefore\ \alpha_1 + 3\alpha_2 = 3$ (A)

$2\alpha_1 + \alpha_2 = -4$ (B)

$\alpha_1 + 5\alpha_2 = 7$ (C)

Now (C) – (A) gives

$2\alpha_2 = 4 \Rightarrow \alpha_2 = 2$

Put in (A), we get $\alpha_1 + 3(2) = 3 \Rightarrow \alpha_1 = -3$

These values satisfy Equation (B) [$\because$ 2 (–3) + 2 = – 4 is true]

So that (1) implies

$L(T) = \{ax + by + c(-3x + 2y) \mid a, b, c \in R\}$

$= \{(a - 3c)x + (b + 2c)y \mid a, b, c \in R\}$

$= \{lx + my \mid l = a - 3c, m = b + 2c \in R\}$

= L (S)

Hence L (T) = L (S).

8. **Prove that every subspace of a finite dimensional vector space has a complement.**

(September 2012)

Sol. Let V(F) be a finite dimensional vector space. Let W_1 be a subspace of V(F), then we have to find a subspace W_2 of V(F) such that $V = W_1 \oplus W_2$.

We know that a subspace of a finite dimensional vector space is finite dimensional.

$\therefore$ W_1 is finite dimensional.

Let $B_1 = \{v_1, v_2, \ldots\ldots, v_m\}$ be a basis of W_1.

Then B_1, being L.I. subset of V, can be extended to form a basis of V.

Let the extended set.

$B_2 = \{v_1, v_2, \ldots\ldots, v_m, w_1, w_2, \ldots\ldots, w_n\}$ be a basis of V.

Let the subspace generated by $\{w_1, w_2, \ldots\ldots, w_n\}$ be denoted by W_2.

To prove that $V = W_1 \oplus W_2$

i.e. to prove $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

Let u be an arbitrary element of V, then u can be expressed as the linear combination of the element of B_2.

i.e. $u = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m + \beta_1 w_1 + \beta_2 w_2 + ... + \beta_n w_n$ for scalars $\alpha_i's$ and $\beta_j's$

$= v + w (say)$

Where $v = \sum_{i=1}^{m} \alpha_i v_i, \quad w = \sum_{j=1}^{n} \beta_j w_j$

$\therefore v \in W_1$ and $w \in W_2$

$\therefore$ each element of V can be expressed as the sum of an element of W_1 and an element of W_2

Thus $V = W_1 + W_2$

Further, let v and w be equal so that

$\sum_{i=1}^{m} \alpha_i v_i = \sum_{j=1}^{n} \beta_j w_j$

$\Rightarrow \sum_{i=1}^{m} \alpha_i v_i + \sum_{j=1}^{n} (-\beta_j) w_j = 0$

$\Rightarrow \alpha_i = 0 \ \forall i$ and $\beta_j = 0 \ \forall j$ [$\because B_2$ is L.I.]

$\Rightarrow \sum_{i=1}^{m} \alpha_i v_i = 0$ and $\sum_{j=1}^{n} \beta_j w_j = 0$

$\Rightarrow v = 0, w = 0$

$\Rightarrow$ no non-zero vector is common to both W_1 and W_2

i.e. $W_1 \cap W_2 = \{0\}$

Hence $V = W_1 \oplus W_2$

9. **Let V(R) be a vector space of all functions from R to R. Show that $V = W_1 \oplus W_2$ where:**
(September 2011)

W_1 = Set of all even functions

W_2 = Set of all odd functions.

Sol. (i) Given $W_1 = \{f \mid f \in V \text{ and } f(-x) = f(x)\}$

Clearly $f(x) = x^4 + x^2 \in W_1$ $[\because f(-x) = f(x)]$

$\therefore W_1$ is non-empty set

Now let $\alpha, \beta \in R$ and $f, g \in W_1$

$\Rightarrow f(-x) = f(x)$ and $g(-x) = g(x)$ for all $x \in R$

$\therefore (\alpha f + \beta g)(-x)$

$= (\alpha f)(-x) + (\beta g)(-x)$

$= \alpha . f(-x) + \beta . g(-x)$

$= \alpha . f(x) + \beta . g(x)$

$= (\alpha f)(x) + (\beta g)(x)$

$= (\alpha f + \beta g)(x)$

$\therefore \alpha f + \beta g \in W_1$

$\Rightarrow W_1$ is a subspace of V.

(ii) Given $W_2 = \{f \mid f \in V \text{ and } f(-x) = -f(x)\}$

Clearly $f(x) = x^3 + x \in W_2$ $[\because f(-x) = -f(x)]$

$\therefore W_2$ is non-empty set

Now let $\alpha, \beta \in R$ and $f, g \in W_2$

$\Rightarrow f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in R$

$\therefore (\alpha f + \beta g)(-x)$

$= (\alpha f)(-x) + (\beta g)(-x)$

$= \alpha . f(-x) + \beta . g(-x)$

$= \alpha(-f(x)) + \beta(-g(x))$

$= -(\alpha . f(x) + \beta . g(x))$

$= -(\alpha f + \beta g)(x)$

$\therefore \alpha f + \beta g \in W_2$

$\Rightarrow W_2$ is a subspace of V.

(iii) To show that $V = W_1 + W_2$.

Let $f \in V$

Then, we can write f as

$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \ \forall x \in R$

$= F(x) + G(x)$, where $F(x) = \frac{1}{2}(f(x) + f(-x))$ $G(x) = \frac{1}{2}(f(x) - f(-x))$

Check that $F(-x) = F(x)$

and $G(-x) = -G(x)$

$\therefore F(x) \in W_1$ and $G(x) \in W_2$

Hence $f = F + G$ where $F \in W_1$ and $G \in W_2$

$\therefore V = W_1 + W_2$.

(iv) Let $f \in W_1 \cap W_2 \Rightarrow f \in W_1$ and $f \in W_2$

$\Rightarrow f(-x) = f(x)$ and $f(-x) = -f(x)$

$\Rightarrow f(x) = -f(x) \quad \Rightarrow \quad 2f(x) = 0$

$\Rightarrow f(x)=0$

So that $W_1 \cap W_2 = \{0\}$

(v) from (iii) and (iv), we get

$V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

$\Rightarrow V = W_1 \oplus W_2$.

Hence the result.

10. If V(F) is a vector space then prove that the set S of non-zero vectors $v_1, v_2, \ldots, v_n \in V$ is linearly dependent if and only if some vector $v_m \in S, 2 \le m \le n$ can be expressed as a linear combination of its preceding vectors.

(September 2011)

Sol. Given $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is a L.D. set

$\exists$ scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$, not all zero, such that

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ (1)

Let k be the largest positive integer for which $\alpha_k \neq 0$

$\therefore \alpha_k \neq 0, \alpha_{k+1} = 0, \alpha_{k+2} = 0, \ldots\ldots, \alpha_n = 0$ (2)

Now we can write (1) as

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k + 0.v_{k+1} + \ldots + 0.v_n = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0$

$$\Rightarrow v_k = \left(-\frac{\alpha_1}{\alpha_k}\right)v_1 + \left(-\frac{\alpha_2}{\alpha_k}\right)v_2 + \ldots\ldots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right)v_{k-1}$$

$\Rightarrow v_k$ can be expressed as the linear combination of its preceding vectors where $2 \le k \le n$.

For if $k = 1$

(2) $\Rightarrow \alpha_1 \neq 0, \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_n = 0$

$\therefore$ (1) $\Rightarrow \alpha_1 v_1 = 0$

$\Rightarrow$ either $\alpha_1 = 0$ or $v_1 = 0$

But neither $\alpha_1 = 0$ nor $v_1 = 0$

$\therefore k \neq 1$

Thus $2 \le k \le n$.

Conversely. Let $v_k, 2 \le k \le n$, be one of the vectors in S which is a linear combination of its preceding vectors,

i.e. $v_k = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_{k-1} v_{k-1}$ for scalars $\beta_1, \beta_2, \ldots\ldots, \beta_{k-1} \in F$

$\Rightarrow \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_{k-1} v_{k-1} + (-1) v_k = 0$

$\Rightarrow$ The set $\{v_1, v_2, \ldots\ldots, v_{k-1}, v_k\}$ is a L.D. set

[$\because \beta_1, \beta_2, \ldots\ldots, \beta_{k-1}, -1$ are not all zero]

$\Rightarrow$ The set $\{v_1, v_2, \ldots\ldots, v_{k-1}, v_k, v_{k+1}, \ldots\ldots, v_n\}$ is a L.D. set.

[$\because$ Every super set of a L.D. set is L.D.]

Hence S is a L.D. set.

11. Let W_1 be a subspace generated by $(-1,2,1), (2,0,1)$ and $(-8,4,-1)$ in $R^3(R)$ and W_2 be the subspace of all vector $(a,0,b)$ for reals a and b. Find basis and dimension of (i) W_1 (ii) W_2 (iii) $W_1 + W_2$. Also find dimension of $W_1 \cap W_2$.

(September 2011)

Sol. (i) To find a basis and dimension of W_1.

Consider a matrix A whose row are the given vectors are reduce it to echelon matrix

$$\text{i.e., } A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ -8 & 4 & -1 \end{bmatrix} \quad \text{Operate } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 8R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & -12 & -9 \end{bmatrix} \quad \text{Operate } R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is an echelon form of A.

Thus non-zero rows of an echelon form of A forms a basis of the row space of matrix A i.e., of subspace W_1

$\therefore B_1 = \{(-1,2,1)(0,4,3)\}$ is a basis of W

and dim $W_1 = 2$.

(ii) To find a basis and dimension of W_2.

Given $W_2 = \{(a,0,b) \mid a,b \in R\}$

These vectors form a subspace of R^3 over R

Now $W_2 = \{(a,0,b) \mid a,b \in R\}$

$= \{(a,0,0) + (0,0,b) \mid a,b \in R\}$

$= \{a(1,0,0) + b(0,0,1) \mid a,b \in R\}$

$\Rightarrow$ each element of W_2 is a linear combination of the vectors

(1, 0, 0) and (0, 0, 1)

The vectors (1, 0, 0) and (0, 0, 1) are L.I. over R since none of vectors is scalar multiple of the other

$\therefore$ (1, 0, 0) and (0, 0, 1) is a basis of W_2

and dim $W_2 = 2$.

(iii) To find a basis and dimension of $W_1 + W_2$.

We know that $W_1 + W_2$ is the space generated by all the vectors of W_1 and W_2.

i.e., the space spanned by the four basis vectors of W_1 and W_2

Consider a matrix P whose rows are the four basis vectors and reduce it to echelon form

$$\text{i.e., } P = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_1 \Leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 - \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 - \frac{1}{2}R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_3 \Leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is an echelon form of P.

Thus non zero rows of an echelon form of matrix P forms a basis of the row space of P i.e., of subspace $W_1 + W_2$.

$\therefore\ B_3 = \{(1,0,0),(0,4,3),(0,0,1)\}$ is a basis of $W_1 + W_2$

and dim $(W_1 + W_2) = 3$

(iv) We know dim $(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim (W_1 + W_2)$

$= 2 + 2 - 3 = 1$

Hence the result.

12. Let $S = \{v_1, v_2, ..., v_n\}$ be a basis for a vector space V over R. Prove that every vector in V can be written as a linear combination of the vectors in S in one and only one way. Is the converse of this statement true? Justify your answer.

(April 2011, April 2010)

Sol. Given $S = \{v_1, v_2, ..., v_n\}$ be a basis of V(F).

$\therefore$ L(S) = V

$\Rightarrow$ any vector $v \in V$ can be expressed as a linear combination of elements of S

i.e. $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n, \alpha_i \in F, 1 \le i \le n$ (1)

Uniqueness:

Let, if possible, $v = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$ (2)

for scalars $\beta_i s \in F$

From (1) and (2),

$\alpha_1 v_1 + \alpha_2 v_2 .... + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$

$\Rightarrow (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + ... + (\alpha_n - \beta_n)v_n = 0$

$\Rightarrow \alpha_1 - \beta_1 = 0,\ \alpha_2 - \beta_2 = 0, ..., \alpha_n - \beta_n = 0$

$[\because v_1, v_2, ..., v_n$ are L.I.$]$

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n$

Hence every vector $v \in V$ can be uniquely expressed as the linear combination of vectors of B.

Conversely. Let each $v \in V$ can uniquely be expressed as

$v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n, \alpha_i \in F, 1 \le i \le n$.

To prove that $S = \{v_1, v_2, ..., v_n\}$ is the basis of V(F).

For this, we have to prove that (i) S is L.I. set

(ii) L(S) = V

(i) S is L.I. set

Consider $\beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n = 0$ for $\beta_i' s \in F$ (3)

Also $0.v_1 + 0.v_2 + ... + 0.v_n = 0$ (4)

Since representation of 0 is unique

$\therefore$ From (3) and (4),

$\beta_1 = 0, \beta_2 = 0, ..., \beta_n = 0$

$\Rightarrow$ S is a L.I. set.

(ii) L(S) = V

Let $v \in V$

$\therefore v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ [given]

$\Rightarrow v \in L(S)$

$\therefore V \subset L(S)$ (5)

Also we know L(S) $\subseteq$ V (6)

From (5) and (6) L(S) = V

Hence S is a basis of V(F)

13. Let S_1, S_2 be finite subsets of a vector space. Let $S_1 \subseteq S_2$. Prove that if S_2 is linearly independent then S_1 is also linearly independent. **(April 2011)**

Sol. Let $S_2 = \{v_1, v_2, ..., v_n\}$ be a L.I. set of vectors.

$\therefore$ for scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$

$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = 0,\ \alpha_2 = 0, ..., \alpha_n = 0$ (1)

Let us consider a subset of S_2 as

$S_1 = \{v_1, v_2, ..., v_k\}$ where $k \le n$

Now $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0$

$\Rightarrow \alpha_1 = 0,\ \alpha_2 = 0, ..., \alpha_k = 0$ $[\because of (1)]$

$\therefore S_1$ is a L.I. set.

14. Let $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$ be a set of vectors in R^4, where $v_1 = (1,2,-2,1)$, $v_2 = (-3,0,-4,3)$, $v_3 = (2,1,1,-1)$, $v_4 = (-3,3,-9,6)$ and $v_5 = (9,3,7,-6)$. Find a subset of S that is a basis for W = Span S. **(April 2011)**

Sol. Given W = Span S

$$S = \begin{bmatrix} 1 & 2 & -2 & 1 \\ -3 & 0 & -4 & 3 \\ 2 & 1 & 1 & -1 \\ -3 & 3 & -9 & 6 \\ 9 & 3 & 7 & -6 \end{bmatrix}$$

operate $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 + 3R_1$, $R_5 \to R_5 - 9R_1$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 6 & -10 & 6 \\ 0 & -3 & 5 & -3 \\ 0 & 9 & -15 & 9 \\ 0 & -15 & 25 & -15 \end{bmatrix}$$

operate $R_3 \to R_3 + \frac{1}{2}R_2$, $R_3 \to R_3 - \frac{3}{2}R_2$, $R_4 \to R_4 + \frac{5}{2}R_2$, $R_2 \to \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence span S is generated by v_1, v_2, and v_3, v_4, v_5 are linear combination of v_1 and v_2

$\Rightarrow$ subset of S such that W = Span S is $\{(1,2,-2,1),(-3,0,-4,3)\}$

15. Let V(R) be a vector space in $R^3(R)$. Examine if the following are subspaces of V or not?

(i) $W = \{(a,b,c) \mid c \text{ is an integer}\}$ **(ii)** $W = \{(a,b,c) \mid a \ge b \ge c\}$.

(September 2010)

Sol. **(i)** Let (a, b, c) $\in$ W where c is an integer

and $\alpha = \sqrt{2} \in R$

Then $\alpha(a,b,c) = (\alpha a, \alpha b, \alpha c)$

$= (\sqrt{2}a, \sqrt{2}b, \sqrt{2}c)$

$\notin$ W [$\because$ c is integer. But $\sqrt{2}$ c is not an integer]

$\therefore$ W is not closed under scalar multiplication.

Hence W is not a subspace of $V_3(R)$.

(ii) Let (a, b, c) $\in$ W, where $a \ge b \ge c$

and $\alpha = -2 \in R$

Then $\alpha(a,b,c) = (\alpha a, \alpha b, \alpha c)$

$= (-2a, -2b, -2c)$

$\notin W$ [Since $a \geq b \geq c \Rightarrow -2a \leq -2b \leq -2c$]

$\therefore$ W is not closed under scalar multiplication.

16. If S is a subset of vector space V(F). Prove that S is a subspace if and only if L(S) = S. (September 2010)

Sol. Let S be a subspace of V(F)

To prove that L(S) = S

Let x ∈ L(S)

Then $x = \sum_{i=0}^{n} \alpha_i x_i$ where $\alpha_i' s \in F$ and $x_i' s \in S$, $1 \leq i \leq n$

Since S is subspace of V(F)

$\therefore$ S is closed w.r.t. vector addition and scalar multiplication.

$\therefore$ x ∈ S

Thus x ∈ L(S) $\Rightarrow$ x ∈ S

$\therefore$ L(S) ⊆ S

Also S ⊆ L(S)

$\therefore$ L(S) = S

Conversely. Let L(S) = S

To prove that S is a subspace of V(F).

We know that L(S) is a subspace of V(F).

Let x, y ∈ L(S)

Then $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$

And $y = \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_m y_m$

Where $\alpha_i \in F, x_i \in S$ for $1 \leq i \leq n$

And $\beta_j \in F, y_j \in S$ for $1 \leq j \leq m$

First, to show that L(S) is a subspace of V(F)

Let $\alpha, \beta \in F$ and x, y ∈ L(S)

Now $\alpha x + \beta y = \alpha(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) + \beta(\beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_m y_m)$

$\Rightarrow \alpha x + \beta y = (\alpha\alpha_1)x_1 + (\alpha\alpha_2)x_2 \ldots + (\alpha\alpha_n)x_n + (\beta\beta_1)y_1 + (\beta\beta_2)y_2 + \ldots + (\beta\beta_m)y_m$

Thus $\alpha x + \beta y$ has been expressed as the linear combination of a finite set $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$ of the elements of S.

$\therefore \alpha x + \beta y \in L(S)$

Thus for x, y ∈ L (S) and $\alpha, \beta \in F$, we have,

$\alpha x + \beta y \in L(S)$

Hence L(S) is a subspace of V(F).

$\therefore$ S is subspace of V [$\because$ L(S) = S]

Hence the proof.

17. Prove that any linearly independent set in V(F) can be extended to a basis of V. (September 2010)

Sol. Let $S = \{w_1, w_2, \ldots, w_k\}$ be a L.I. subset of a finite dimensional vector space V(F).

If dim V = n, then V has a finite basis $B = \{v_1, v_2, \ldots, v_n\}$, so that B is L.I. and L(B) = V

Consider a set $S_1 = \{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_n\}$

Now since each v ∈ V, can be expressed as the linear combination of the elements of B,

$\therefore$ v ∈ V

$\Rightarrow v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, $\alpha_i' s \in F$

$\Rightarrow v = 0.w_1 + 0.w_2 + \ldots + 0.w_k + \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

$\Rightarrow$ v can be expressed as the linear combination of elements of S_1

$\Rightarrow v \in L(S_1)$

$\therefore V \subset L(S_1)$

Also $L(S_1) \subset V$

$\therefore L(S_1) = V$.

Also since we can express all $w_j' s$ as linear combination of $v_i' s$

$\therefore$ S_1 is L.D. set.

$\Rightarrow$ there is some vector of S_1 which is a linear combination of its preceding vectors.

This vector cannot be any of $w_j' s$ [$\because w_j' s$ are L.I.]

$\therefore$ this vector must be some v_k

Consider a set $S_2 = \{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$

Obviously $L(S_2) = V$

If S_2 is L.I. set then S_2 is a basis of V and it is the required extended set, which is a basis of V.

If S_2 is L.D. then repeating the above process a finite number of times till we get a L.I. set which contains S and spanning V. This set will be the extension of set S and will be a basis of V.

Hence any L.I. set in V can be extended to a basis of V.

18. Let M and N are two subspaces of R^4: where $M=\{(a,b,c,d)\,|\,b+c+d=0\}$ $N=\{(a,b,c,d)\,|\,a+b=0,c=2d\}$ Find a basis and dimension of:

(i) M (ii) N (iii) M ∩ N. (September 2010)

Sol. (i) $M=\{(a,b,c,d):b+c+d=0\}$

We seek a basis of the set of solutions (a, b, c, d) of the equation b + c + d = 0

Now $b=-c-d$.

Then $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}a+\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}c+\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}d$

Here a, c, d are independent variables.

∴ basis of $M=\{(1,0,0,0),(0,-1,1,0),(0,-1,0,1)\}$

and dim M = 3.

(ii) $N=\{(a,b,c,d):a+b=0,c=2d\}$

We seek a basis of the set of solutions (a, b, c, d) of the system of equations

a + b = 0, c = 2d

i.e. a = –b, c = 2d

Now $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}b+\begin{bmatrix}0\\0\\2\\1\end{bmatrix}d$

Here b, d are independent variables.

∴ basis of $N=\{(-1,1,0,0),(0,0,2,1)\}$

and dim N = 2

(iii) $M\cap N=\{(a,b,c,d):b+c+d=0 \text{ and } a+b=0,c=2d\}$

We seek a basis of the set of solutions (a, b, c, d) of the system of equations b + c + d = 0, a + b = 0, c = 2d

i.e. $b=-3d$, $a=3d$, $c=2d$

Now $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}3\\-3\\2\\1\end{bmatrix}d$

Here d is the independent variable.

∴ basis of $M\cap N=\{(3,-3,2,1)\}$

and dim $(W_1\cap W_2)=1$

19. Prove that the necessary and sufficient condition for non-empty subset W of a vector space V(F) to be subspace of V is that $\alpha x+\beta y$ for α, $\beta\in F$ and $x,y\in W$

(April 2010, September 2009)

Sol. Necessary condition.

Given : W is a subspace of V(F)

To prove : $\alpha x+\beta y\in W\ \forall\alpha,\beta\in F$ and x, y ∈ W

Since W is a subspace of V(F)

∴ W is closed under addition and scalar multiplication.

Now $\alpha\in F, x\in W \quad\Rightarrow \alpha x\in W$

and $\beta\in F, y\in W \quad\Rightarrow \beta y\in W$

Also $\alpha x\in W, \beta y\in W \quad\Rightarrow \alpha x+\beta y\in W$

Sufficient condition.

Given : $\alpha x+\beta y\in W\ \forall\alpha,\beta\in F$ and x, y ∈ W

To prove: W is a subspace of V(F)

Taking $\alpha=1$, $\beta=-1$, we have

$\alpha x+\beta y\in W\Rightarrow x-y\in W\ \forall x,y\in W$

Again taking $\beta=0$, we have,

$\alpha x+\beta y\in W\Rightarrow \alpha x\in W\ \forall x\in W$

Now since $x-y\in W \quad \forall x,y\in W$

And $\alpha x\in W \quad \forall\alpha\in F, x\in W$

∴ W is a subspace of V(F).

20. Find the condition on a, b, c so that the vector $v=(a,b,c)\in R^3$ belongs to the space generated by $v_1=(2,1,0)$, $v_2=(1,-1,2)$ and $v_3=(0,3,-4)$. (April 2010)

Sol. Let $v=\alpha v_1+\beta v_2+\gamma v_3$ for some $\alpha,\beta,\gamma\in R$.

$\Rightarrow (a,b,c)=\alpha(2,1,0)+\beta(1,-1,2)+\gamma(0,3,-4)$

$=(2\alpha,\alpha,0)+(\beta,-\beta,2\beta)+(0,3\gamma,-4\gamma)$

$=(2\alpha+\beta,\alpha-\beta+3\gamma,2\beta-4\gamma)$

$\Rightarrow 2\alpha+\beta=a$ (1)

$\alpha-\beta+3\gamma=b$ (2)

$2\beta-4\gamma=c$ (3)

From (1), $\alpha=\dfrac{a-\beta}{2}$

From (3), $\gamma=\dfrac{2\beta-c}{4}$

$\therefore (2) \Rightarrow \frac{a-\beta}{2} - \beta + \frac{6\beta - 3c}{4} = b$

$\Rightarrow 2a - 2\beta - 4\beta + 6\beta - 3c = 4b$

$\Rightarrow 2a - 4b - 3c = 0$

$\therefore$ v belongs to the space generated by v_1, v_2, v_3

If $2a - 4b - 3c = 0$

21. Find a basis and dimension of the subspace W generated by the vectors $(1,-1,1),(8,4,2),(2,2,0),(3,9,-3)$ of R^3. Also extend this basis to a basis of R^3.

(April 2010)

Sol. Consider a matrix A whose rows are the given vectors and reduce it to echelon matrix.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 8 & 4 & 2 \\ 2 & 2 & 0 \\ 3 & 9 & -3 \end{bmatrix}$$

Operate $R_2 \to R_2 - 8R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 12 & -6 \\ 0 & 4 & -2 \\ 0 & 12 & -6 \end{bmatrix}$$

Operate $R_3 \to R_3 - \frac{1}{3}R_2, R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 12 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Operate $R_2 \to \frac{1}{6}R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is an echelon form of A.

Thus non-zero rows of an echelon form of A, forms a basis of the row space of A i.e., of subspace W of R^3.

Hence $B = \{(1,-1,1),(0,2,-1)\}$ is a basis of W, so that dim W = 2.

IInd part. Since dim $R^3 = 3$

$\therefore$ We need three L.I. vectors of R^3 over R which include the above vector (1, −1, 1) and (0, 2, −1).

Consider $B_1 = \{(1,-1,1),(0,2,-1),(0,0,1)\}$

To show B_1 is L.I.

From the matrix, whose rows are the vectors of B_1.

i.e. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ which is an echelon matrix having all the three non-zero rows.

$\therefore$ B_1 is L.I.

Hence B_1 is a basis of R^3, which is an extension of the basis of W.

22. For what value of k, will the vector $v = (1,k,-4)$ be a linear combination of $v_1 = (1,-3,2)$ and $v_2 = (2,-1,1)$. **(September 2009)**

Sol. Let $v = \alpha v_1 + \beta v_2$

$(1,k,-4) = \alpha(1,-3,2) + \beta(2,-1,1)$

$= (\alpha + 2\beta, -3\alpha - \beta, 2\alpha + \beta)$

$\Rightarrow \alpha + 2\beta = 1$ (1)

$-3\alpha - \beta = k$ (2)

$2\alpha + \beta = -4$ (3)

$\Rightarrow 2(1) - (3) \Rightarrow 2(\alpha + 2\beta) - (2\alpha + \beta) = 2 - (-4)$

$\Rightarrow 3\beta = 6 \Rightarrow \beta = 2$

From (1) $\alpha + 2(2) = 1 \Rightarrow \alpha = -3$

Hence $k = -3\alpha - \beta = -3(-3) - (2)$

$= 9 - 2 = 7$

$\Rightarrow k = 7$

23. Find a basis and dimension of subspace W of R^4 generated by $(1,-2,5,-3),(2,3,1,-4),(3,8,-3,-5)$. Also extend this to basis of R^4.

(September 2009)

Sol. (a) Consider a matrix A whose rows are the given vectors and reduce it to echelon matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

Operate $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \quad \text{Operate } R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is an echelon form of A.

Thus non zero rows of an echelon form of A, forms a basis of the row space of A i.e., of subspace W of R^4.

Hence $B = \{(1,-2,5,-3),(0,7,-9,2)\}$ is a basis of W, so that dim W = 2

IInd – Part. Since dim $R^4 = 4$

$\therefore$ We need four L.I. vectors of R^4 over R which include the above vectors $(1,-2,5,-3),(0,7,-9,2)$

Consider $B_1 = \{(1,-2,5,-3),(0,7,-9,2),(0,0,1,0),(0,0,0,1)\}$

To show B_1 is L.I.

From the matrix, whose rows are the vectors of B_1

i.e.,
$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is an echelon matrix having all the four non-zero rows.

$\therefore$ B_1 is L.I. set

Hence B_1 is a basis of R^4, which is an extension of the basis of W.

24. Let W_1 and W_2 be the subspace of R^4 generated by:

$\{(1,2,2,-2),(2,3,2,-3)(1,3,4,-3)\}$ and $\{(1,1,0,-1),(1,2,3,0),(6,9,9,-3)\}$

respectively. Find a basis and dimension of

(i) W_1 (ii) W_2 (iii) $W_1 \cap W_2$. **(April 2009)**

Sol. (i) To find a basis and dimension of W_1

Consider a matrix A whose rows are the given vectors and reduce it to echelon matrix

$$\text{i.e., } A = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \quad \text{Operate } \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \quad \text{Operate } R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is an echelon form of A.

The non-zero rows of an echelon form of A forms a basis of row space of A i.e., of subspace W_1

Thus $B_1 = \{(1,2,2,-2),(0,-1,-2,1)\}$ is a basis of W_1

and dim $W_1 = 2$.

(ii)To find a basis and dimension of W_2. Consider matrix B whose rows are the given vectors and reduce it to echelon matrix

$$\text{i.e., } B = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 6 & 9 & 9 & -3 \end{bmatrix} \quad \text{Operate } \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 9 & 3 \end{bmatrix} \quad \text{Operate } R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is an echelon form of B.

Thus non-zero rows of an echelon form of matrix B forms a basis of row space of B i.e., of subspace W_2

$\therefore$ $B_2 = \{(1, 1, 0, -1), (0, 1, 3, 1)\}$ is a basis of W_2

and dim $W_2 = 2$

(iii) To find a basis and dimension of $W_1 + W_2$

We know that $W_1 + W_2$ is the space generated by all the vectors of W_1 and W_2

i.e., the space generated by the basis vectors of W_1 and W_2

Consider a matrix C, whose rows are the four basis vectors and reduce it to echelon matrix i.e., $C = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$ Operate $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$ Operate $R_3 \to R_3 - R_2$

$R_4 \to R_4 + R_2$

$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Operate $R_3 \Leftrightarrow R_4$

$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is an echelon form of C.

Thus non-zero rows of an echelon form of matrix C forms a basis of row space of C i.e., of subspace $W_1 + W_2$

$\therefore B_3 = \{(1,2,2,-2),(0,-1,-2,1),(0,0,1,2)\}$ is a basis of $W_1 + W_2$

and $\dim(W_1 + W_2) = 3$.

(iv) We know that

$\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2)$

$= 2 + 2 - 3 = 1$

Hence the result.

4

LINEAR TRANSFORMATIONS

1. Let T be a linear operator on R^3 defined by **(September 2013)**

$T(x,y,z) = (2x,\ 4x-y,\ 2x+3y-z)$

Prove that T is invertible and find T^{-1}.

Sol. We know that T is invertible iff T is one-one and onto

(i) To prove that T is one-one

Let $v_1 = (x_1,y_1,z_1)$ and $v_2 = (x_2,y_2,z_2) \in V_3(R)$

Such that $T(v_1) = T(v_2)$

$\Rightarrow T(x_1,y_1,z_1) = T(x_2,y_2,z_2)$

$\Rightarrow (2x_1,\ 4x_1 - y_1,\ 2x_1 + 3y_1 - z_1) = (2x_2,\ 4x_2 - y_2,\ 2x_2 + 3y_2 - z_2)$

$\Rightarrow 2x_1 = 2x_2 \qquad \Rightarrow x_1 = x_2$

$4x_1 - y_1 = 4x_2 - y_2 \qquad \Rightarrow y_1 = y_2 \qquad [\because x_1 = x_2]$

and $2x_1 + 3y_1 - z_1 = 2x_2 + 3y_2 - z_2 \Rightarrow z_1 = z_2 \qquad [\because x_1 = x_2 \text{ and } y_1 = y_2]$

$\therefore (x_1,y_1,z_1) = (x_2,y_2,z_2)$

$\Rightarrow v_1 = v_2$

$\therefore T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

$\therefore$ T is one-one

(ii) To prove T is onto: Let $(a,b,c) \in V_3(R)$ and we will show that $\exists$ a vector $(x,y,z) \in V_3(R)$ such that

$T(x,y,z) = (a,b,c)$

$\Rightarrow (2x,\ 4x-y,\ 2x+3y-z) = (a,b,c)$

$\Rightarrow 2x = a,\ 4x - y = b,\ 2x + 3y - z = c$

$\Rightarrow x = \frac{a}{2},\ y = 2a - b,\ z = 7a - 3b - c$

Since $a, b, c \in R \Rightarrow x, y, z \in R$

$\therefore (x, y, z) = \left(\frac{a}{2},\ 2a - b,\ 7a - 3b - c\right) \in V_3(R)$

Thus T is onto

Hence T is one-one and onto

$\Rightarrow$ T is invertible

$\therefore T(x, y, z) = (a, b, c)$

$\Rightarrow T^{-1}(a, b, c) = (x, y, z) = \left(\frac{a}{2},\ 2a - b,\ 7a - 3b - c\right)$

$\Rightarrow T^{-1}(x, y, z) = \left(\frac{x}{2},\ 2x - y,\ 7x - 3y - z\right)$ is the required inverse of T.

2. **Prove that a linear transformation $T : V \to W$ is non singular iff set of images of a linearly independent set is linearly independent.** **(September 2013)**

Sol. Given $T : V \to W$ is a non-singular L.T.

Let $S = \{v_1, v_2, ... v_n\}$ be a L.I. subset of V

To prove: $S' = \{T(v_1), T(v_2), ... T(v_n)\}$ is L.I. set

Consider $\alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_n T(v_n) = 0$ for $\alpha_i' s \in F$ (1)

$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + ... + T(\alpha_n v_n) = 0$ [$\because$ T is a L.T.]

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n) = 0$ [$\because$ T is a L.T.]

$\Rightarrow \alpha_1 v_1 - \alpha_2 v_2 + ... + \alpha_n v_n = 0$ [$\because$ T is a non-singular]

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, ..., \alpha_n = 0$ [$\because$ S is a L.I. set]

Thus (1) $\Rightarrow S'$ is a L.I. set

Conversely. Given: The set of images of L.I. set is L.I.

To prove: T : V $\to$ W is non-singular

Let $v \in V$ be such that T(v) = 0

If possible, let us suppose v $\neq$ 0

$\Rightarrow$ {v} is a L.I. set [Singleton non zero vector in a vector space is L.I.]

$\Rightarrow$ {T(v)} is also L.I. set [given]

$\Rightarrow$ {0} is a L.I. set

which is a contradiction [$\because$ singleton zero set is L.D.]

$\therefore$ our supposition is wrong

$\Rightarrow$ v = 0

Thus T(v) = 0 $\Rightarrow$ v = 0 for $v \in V$

$\Rightarrow$ T : V $\to$ W is non-singular

3. **Let the linear transformation $T : R^3 \to R^3$ be defined by $T(x, y, z) = (2x,\ 4x - y,\ 2x + 3y - z)$ Verify Rank–Nullity Theorem for T.**

(September 2013)

Sol. Firstly we shall find basis for range T

Since B is a basis of R^3

$\therefore B_1 = \{T(e_1), T(e_2), T(e_3)\}$ generates range T

Here $T(e_1) = T(1,0,0) = (2, 4-0, 2+0-0) = (2,4,2)$

$T(e_2) = T(0,1,0) = (0, 0-1, 0+3-0) = (0,-1,3)$

$T(e_3) = T(0,0,1) = (0, 0-0, 0+0-1) = (0,0,-1)$

$\therefore B_1 = \{(2,4,2), (0,-1,3), (0,0,-1)\}$ generates range T

Consider the matrix $\begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

which is in echelon form and have three non-zero rows and form L.I. set of vectors

$\therefore$ Range space of $T = \{(2,4,2), (0,-1,3), (0,0,-1)\}$

Rank T = Number of elements in this basis = 3

To find basis for Null space of T

Let $v = (x, y, z) \in N(T)$

$T(v) = T(x, y, z) = 0$

$\Rightarrow (2x,\ 4x - y,\ 2x + 3y - z) = (0,0,0)$

$\Rightarrow 2x = 0,\ 4x - y = 0,\ 2x + 3y - z = 0$

$\Rightarrow x = 0,\ y = 0,\ z = 0$

$\Rightarrow (x, y, z) = (0,0,0)$

$\Rightarrow v = (0,0,0)$

So that $v \in N(T) \Rightarrow v = (0,0,0) = 0$

$\therefore$ Null space of $T = \{0\}$

and Nullity $T = \dim N(T) = 0$

Thus Nullity T + Rank T = 0 + 3 = 3 = $\dim R^3$

Hence the result.

4. State and prove rank nullity theorem. **(April 2013, September 2011)**

Sol. **Statement:** If T: V → W be a linear transformation from a finite dimensional vector space V(F) to W(F) then
dimV = Rank T + Nullity T

Let dim V = n

Since Null space of T is a sub space of V

∴ it is finite dimensional

Let $\dim(N(T)) = k \le n$

∴ Nullity T = k

⇒ Basis set of N(T) contains k elements

Let $B_1 = \{v_1, v_2, ..., v_k\}$ be the basis set of N(T).

By definition of N(T), $T(v_1) = 0$, $T(v_2) = 0, ...T(v_k) = 0$

Now, the set B_1 can be extended so that the extended set

$B_2 = \{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$ consisting of n elements, is a basis set of V.

We consider a set B_3,

$B_3 = \{T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$

The set B_3 consists of T images of (n – k) vectors.

If we prove that B_3 is a basis for R(T), the theorem will be proved.

We have to show that:

(i) B_3 is L.I. set **(ii) B_3 spans R(T)** **(iii) B_3 is L.I.**

Let $\alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n \in F$ such that

$\alpha_{k+1}T(v_{k+1}) + \alpha_{k+2}T(v_{k+2}) + ... + \alpha_n T(v_n) = 0$

$\Rightarrow T(\alpha_{k+1}v_{k+1} + \alpha_{k+2}v_{k+2} + ... + a_n v_n) = 0$ [∵ T is a L.T.]

$\Rightarrow \alpha_{k+1}v_{k+1} + \alpha_{k+2}v_{k+2} + ... + \alpha_n v_n \in N(T)$

Now since B_1 is a basis of N(T)

∴ every element of N(T) can be written as the linear combination of elements of B_1

$\Rightarrow \alpha_{k+1}v_{k+1} + \alpha_{k+2}v_{k+2} + ... + \alpha_n v_n = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k + (-\alpha_{k+1})v_{k+1} + (-\alpha_{k+2})v_{k+2} ..... + (-\alpha_n)v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_k = \alpha_{k+1} = \alpha_{k+2} = ... = \alpha_n = 0$ [∵ B_2 is a L.I. set]

$\therefore \alpha_{k+1}T(v_{k+1}) + \alpha_{k+2}T(v_{k+2}) + ... + \alpha_n T(v_n) = 0$

$\Rightarrow \alpha_{k+1} = \alpha_{k+2} = ... = \alpha_n = 0$

⇒ B_3 is a L.I. set.

(ii) B_3 spans R(T)

Let y be any element of R(T)

∴∃ x ∈ V such that T(x) = y

Also ∵ x ∈ V and B_2 is the basis set of V

$\therefore x = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_k v_k + \beta_{k+1}v_{k+1} + ... + \beta_n v_n$

$\Rightarrow y = T(x) = T(\beta_1 v_1 + \beta_2 v_2 + ... + \beta_k v_k + \beta_{k+1}v_{k+1} + ... + \beta_n v_n)$

$= \beta_1 T(v_1) + \beta_2 T(v_2) + ... + \beta_k T(v_k) + \beta_{k+1}T(v_{k+1}) + ... + \beta_n T(v_n)$

$= \beta_1 .0 + \beta_2 .0 + ... + \beta_k .0 + \beta_{k+1}T(v_{k+1}) + ... + \beta_n T(v_n)$

$= \beta_{k+1}T(v_{k+1}) + ... + \beta_n T(v_n)$

⇒ y ∈ R(T) is a linear combination of elements of B_3

∴ B_3 spans R(T)

Hence B_3 is a basis of R(T)

⇒ dim R(T)) = n – k

= dim V – dim N(T)

⇒ Rank T + Nullity T = dim V

5. Show that U_1, U_2 are subspaces of R^4: **(April 2013)**

$U_1 = \{(a,b,c,d);\ b + c + d = 0\}$

$U_2 = \{(a,b,c,d);\ a + b = 0,\ c = 2d\}$

Sol. **(i)** Let $\alpha_1, \alpha_2 \in U_1$, where $\alpha_1 = (a_1, b_1, c_1, d_1)$

and $\alpha_2 = (a_2, b_2, c_2, d_2)$ with $b_1 + c_1 + d_1 = 0$

and $b_2 + c_2 + d_2 = 0$

Let x, y ∈ R

Now $x\alpha_1 + y\alpha_2 = x(a_1, b_1, c_1, d_1) + y(a_2, b_2, c_2, d_2)$

$= (xa_1 + ya_2, xb_1 + yb_2, xc_1 + yc_2, xd_1 + yd_2)$

Consider $xb_1 + yb_2 + xc_1 + yc_2 + xd_1 + xd_2 = x(b_1 + c_1 + d_1)$

$+ y(b_2 + c_2 + d_2) = x(0) + y(0) = 0$

$\Rightarrow x\alpha_1 + y\alpha_2 \in U_1$ Hence U_1 is subspace of R^4

(ii) Let $\beta_1, \beta_2 \in U_2$ where $\beta_1 = (a_1, b_1, c_1, d_1)$ and $\beta_2 = (a_2, b_2, c_2, d_2)$ with $a_1 + b_1 = 0$, $a_2 + b_2 = 0$, $c_1 = 2d_1$ and $c_2 = 2d_2$

Now $x\beta_1 + y\beta_2 = x(a_1, b_1, c_1, d_1) + y(a_2, b_2, c_2, d_2)$

$= (xa_1 + ya_2,\ xb_1 + yb_2,\ xc_1 + yc_2,\ xd_1 + xd_2)$

Consider $xa_1 + ya_2 + xb_1 + yb_2 = x(a_1 + b_1) + y(a_2 + b_2) = x(0) + y(0) = 0$

Also $xc_1 + yc_2 = x(2d_1) + y(2d_2) = 2(xd_1 + yd_2)$

$\Rightarrow x\beta_1 + y\beta_2 \in U_2$. Hence U_2 is subspace of R^4

6. **Let T be a linear operator on R^3 defined by:** **(September 2012)**

$T(x,y,z)=(x-2y-z,y-zx)$

Show that T is invertible and find T^{-1}.

Sol. We know that T is invertible iff T is non singular

To show T is non-singular

Let $T(x,y,z)=(0,0,0)$ for $(x,y,z)\in R^3$

$\Rightarrow (x-2y-z,y-z,x)=(0,0,0)$

$\Rightarrow x-2y-z=0,\ y-z=0$ and $x=0$

$\Rightarrow -2y-z=0, y-z=0$

$\Rightarrow x=0, y=0, z=0$

$\Rightarrow (x,y,z)=(0,0,0)$

$\therefore T(x,y,z)=(0,0,0) \quad \Rightarrow (x,y,z)=(0,0,0)$

$\Rightarrow$ T is non-singular

$\Rightarrow$ T is invertible operator on R^3.

To find T^{-1}

Let $T(x,y,z)=(a,b,c)$

$\Rightarrow (x-2y-z,y-z,x)(a,b,c)$

$\Rightarrow x-2y-z=a, y-z=b, x=c$

Solving $x=c, y=\dfrac{-a+b+c}{3}, z=\dfrac{-a-2b+c}{3}$

Thus T^{-1} is given by

$T^{-1}(a,b,c)=(x,y,z)$

$$\Rightarrow T^{-1}(a,b,c)=\left(c,\frac{-a+b+c}{3},\frac{-a-2b+c}{3}\right)$$

7. **Let T be a linear operator on R^3 defined by: $T(x,y,z)=(2y+z,x-4y,3x)$**
Find the matrix of T relative to the basis $\{(1,1,1),(1,1,0),(1,0,0)\}$.

(September 2012)

Sol. Firstly, we shall express any element

$v=(\alpha,\beta,\gamma)\in R^3$ as a linear combination of the elements of basis B

Let $(\alpha,\beta,\gamma)=a(1,1,1)+b(1,1,0)+c(1,0,0)$ for some reals a, b, c

$=(a+b+c,a+b,a)$

$\Rightarrow a+b+c=\alpha,\ a+b=\beta,\ a=\gamma$

Solving these, we get, $a=\gamma, b=\beta-\gamma, c=\alpha-\beta$

$\therefore (\alpha,\beta,\gamma)=\gamma(1,1,1)+(\beta-\gamma)(1,1,0)+(\alpha-\beta)(1,0,0) \quad (1)$

Given $T:R^3\to R^3$ is a linear operator defined as

$T(x,y,z)=(2y+z,x-4y,3x)$

and $B=\{(1,1,1),(1,1,0),(1,0,0)\}$ is a basis of R^3

Now $T(1,1,1)=(2+1,1-4,3)=(3,-3,3)$

$=3(1,1,1)+(-3-3)(1,1,0)+(3+3)(1,0,0)$ [Using (1)]

$=3(1,1,1)+(-6)(1,1,0)+(6)(1,0,0)$

$T(1,1,0)=(2+0,1-4,3)=(2,-3,3)$

$=3(1,1,1)+(-3-3)(1,1,0)+(2+3)(1,0,0)$ [Using (1)]

$=3(1,1,1)+(-6)(1,1,0)+(5)(1,0,0)$

$T(1,0,0)=(0+0,1-0,3)=(0,1,3)$

$=3(1,1,1)+(1-3)(1,1,0)+(0-1)(1,0,0)$ [Using (1)]

$=3(1,1,1)+(-2)(1,1,0)+(-1)(1,0,0)$

$$\therefore [T;B]=\begin{bmatrix}3&-6&6\\3&-6&5\\3&-2&-1\end{bmatrix}'=\begin{bmatrix}3&3&3\\-6&-6&-2\\6&5&-1\end{bmatrix}$$

8. **State and prove the first isomorphism theorem of linear transformations.**

(April 2012)

Sol. **First Isomorphism theorem:** Each linear transformation $T:V\to W$ induces a linear isomorphism $\quad \phi:\dfrac{v}{\ker T}\to rankT$

Proof: Let $\phi:\dfrac{v}{\ker T}\to rankT$ be defined by $\phi([v])=Tv$, for all $[v]\in\dfrac{v}{\ker T}$.

<u>**ϕ is well defined:**</u>The value of ϕ on a coset [v] does not depend on the particular representative v for that coset. Let $v,v'\in V$ be such that

$[v]=[v']\Rightarrow[v-v']=[0]\Rightarrow v-v'\in KerT$

$\Rightarrow T(v-v')=0\Rightarrow Tv=Tv'$

$\Rightarrow\phi$ is well defined.

<u>**ϕ is linear transformation:**</u>

$\phi(\alpha[v]+\beta[v'])=\phi([\alpha v+\beta v'])=T(\alpha v+\beta v')$

$=\alpha T(v)+\beta T(v')=\alpha\phi([v])+\beta\phi([v'])$

$\Rightarrow\phi$ is L.T.

ϕ is surjective:

If $w \in$ rank T then there exists some $v \in V$ for which w = Tv, so $w = \phi([v])$

ϕ is injective:

Suppose $\phi([v_1]) = \phi([v_2]) \Rightarrow Tv_1 = Tv_2$

$\Rightarrow T(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 \in KerT$

$\Rightarrow [v_1] = [v_2]$

9. **Prove that a linear transformation T : V → W is non-singular if and only if set of images of linearly independent set is linearly independent.** **(April 2012, 2010)**

Sol. Given $T: V \to W$ is a non-singular L.T.

Let $S = \{v_1, v_2, \ldots\ldots v_n\}$ be a L.I. subset of V

To prove: $S' = \{T(v_1), T(v_2), \ldots\ldots T(v_n)\}$ is L.I. set

Consider $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n) = 0$ for $\alpha_i's \in F$ (1)

$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \ldots + T(\alpha_n v_n) = 0$ [∵ T is a L.T.]

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = 0$ [∵ T is a L.T.]

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ [∵ T is a non-singular]

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$ [∵ S is a L.I. set]

Thus (1) $\Rightarrow S'$ is a L.I. set

Conversely. Given: The set of images of L.I. set is L.I.

To prove: T : V → W is non-singular

Let $v \in V$ be such that T(v) = 0

If possible, let us suppose $v \neq 0$

$\Rightarrow$ {v} is a L.I. set [Singleton non zero vector in a vector space is L.I.]

$\Rightarrow$ {T(v)} is also L.I. set [given]

$\Rightarrow$ {0} is a L.I. set

which is a contradiction [∵ singleton zero set is L.D.]

∴ our supposition is wrong

$\Rightarrow$ v = 0

Thus T(v) = 0 $\Rightarrow$ v = 0 for $v \in V$

$\Rightarrow$ T : V → W is non-singular

10. **Let V be a finite dimensional vector space over the field F, and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Then show that there is a unique dual $B^* = \{f_1, f_2, \ldots, f_n\}$ for the dual V* of V such that $f_i(a_j) = \delta_{ij}$. Also show that for each linear functional f on V, $f = \sum_{i=1}^{n} f(\alpha_i) f_i$ and for each vector α in V, $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.** **(April 2012)**

Sol. $B = \{\alpha_1, \ldots\ldots, \alpha_n\}$ is an ordered basis for V.

Therefore there exists a unique linear functional f_1 on V such that

$f_1(\alpha_1) = 1, f_1(\alpha_2) = 0, \ldots\ldots, f_1(\alpha_n) = 0$

Where {1, 0, ……, 0} is an ordered set of n scalars.

In fact, for each $i = 1, 2, \ldots n$ there exists a unique linear functional f_i on V such that

$$f_i(\alpha_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

i.e. $f_i(x_j) = \delta_{ij}$, (1)

Where $\delta_{ij} \in F$ is Kronecker delta i.e. $\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$ if $i \neq j$

Let $B^* = \{f_1, \ldots\ldots, f_n\}$. Then B^* is a subset of V^* containing n distinct elements of V^*. We shall show that B^* is a basis for V^*.

First we shall show that B^* is linearly independent.

Let $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = \hat{0}$

$\Rightarrow (c_1 f_1 + c_2 f_2 + \ldots + c_n f_n)\alpha = \hat{0}(\alpha) \quad \forall \alpha \in V$

$\Rightarrow c_1 f_1(\alpha) + \ldots + c_n f_n(\alpha) = 0 \quad \forall \alpha \in V \quad [\because \hat{0}(\alpha) = 0]$

$\Rightarrow \sum_{i=1}^{n} c_i f_i(\alpha) = 0 \quad \forall \alpha \in V$

$\Rightarrow \sum_{i=1}^{n} c_i f_i(\alpha_j) = 0, j = 1, 2, \ldots\ldots, n$ [Putting $\alpha = \alpha_j$ where j = 1, 2,……, n]

$\Rightarrow \sum_{i=1}^{n} c_i \delta_{ij} = 0, j = 1, 2, \ldots\ldots, n$

$\Rightarrow c_j = 0, j = 1, 2, \ldots, n$

$\Rightarrow f_1, f_2, \ldots\ldots, f_n$ are linearly independent.

In the second place, we shall show that the linear span of B^* is equal to V^*.

Let f be any element of V^*. The linear functional f will be completely determined if we define it on a basis for V. So let

$f(\alpha_i) = a_i, i = 1, 2, ..., n$ (2)

We shall show that $f = a_1 f_1 + ... + a_n f_n = \sum_{i=1}^{n} a_i f_i$

We know that two linear functionals on V are equal if they agree on a basis of V. So let $\alpha_j \in B$ where j = 1,..........,n. Then

$\left[\sum_{i=1}^{n} a_i f_i\right](\alpha_j) = \sum_{i=1}^{n} a_i f_i(\alpha_j)$

$= \sum_{i=1}^{n} a_i \delta_{ij}$ [From (1)]

$= a_j$ on summing with respect to i and remembering that $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$

$= f(\alpha_j)$ [From (2)]

Thus $\left[\sum_{i=1}^{n} a_i f_i\right](\alpha_j) = f(\alpha_j) \quad \forall \alpha_j \in B$.

Therefore $f = \sum_{i=1}^{n} a_i f_i$. Thus every element f in V^* can be expressed as a linear combination of $f_1, ..., f_n$.

$\therefore$ V^* = linear span of B^*. Hence B^* is a basis for V^*.

Now dim V^* = number of distinct elements of $B^* = n$.

Since B^* is dual basis of B, therefore

$f_i(\alpha_j) = \delta_{ij}$. (3)

If f is a linear functional on V, then $f \in V^*$ for which B^* is basis. Therefore f can be expressed as a linear combination of $f_1, ..., f_n$. Let $f = \sum_{i=1}^{n} c_i f_i$.

Then $f(\alpha_j) = \left(\sum_{i=1}^{n} c_i f_i\right)(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j)$

$= \sum_{i=1}^{n} c_i \delta_{ij}$ [From (1)]

$= c_j, j = 1, 2, ..., n$.

$\therefore f = \sum_{i=1}^{n} f(\alpha_i) f_i$.

Now let α be any vector in V. Let

$\alpha = x_1\alpha_1 + ... + x_n\alpha_n$. (4)

Then $f_i(\alpha) = f_i\left(\sum_{j=1}^{n} x_j \alpha_j\right)$ $\left[\text{from (2), } \alpha = \sum_{j=1}^{n} x_j \alpha_j\right]$

$= \sum_{j=1}^{n} x_j f_i(\alpha_j)$ [$\because f_i$ is linear functional]

$= \sum_{j=1}^{n} x_j \delta_{ij}$ [From (3)]

$= x_i$.

$\therefore \alpha = f_1(\alpha)\alpha_1 + ... + f_n(\alpha)\alpha_n = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.

11. **Let $L : R^4 \to R^3$ be defined by $L(x, y, z, w) = (x + y, y - z, z - w)$. Verify Rank-Nullity theorem for L.** **(April 2011)**

Sol. We know basis of R^4 is

$B = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\} = \{e_1, e_2, e_3, e_4\}$

$B' = \{T(e_1), T(e_2), T(e_3), T(e_4)\}$ generates range T

$T(e_1) = T(1,0,0,0) = (1,0,0)$, $T(e_2) = T(0,1,0,0) = (1,1,0)$

$T(e_3) = T(0,0,1,0) = (0,-1,1)$, $T(e_4) = T(0,0,0,1) = (0,0,-1)$

Consider the matrix = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Operate $R_2 \to R_2 - R_1 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Operate $R_3 \to R_3 + R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Operate $R_4 \to R_4 + R_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence Range $T = \{(1,0,0),(0,1,0),(0,0,1)\} \Rightarrow R(T) = 3$

To find Null space of T

$T(x,y,z,w) = 0 \Rightarrow (x+y,\ y-z,\ z-w) = 0$

$\Rightarrow\ x = -y,\ y = z,\ w = z \Rightarrow x = -z,\ y = z,\ w = z$

$\Rightarrow \{(-1,1,1,1)\}$ is Null space of T $\Rightarrow$ N(T) = 1

$\therefore$ Nullity T + Rank T = 1 + 3 = 4 = dim R^4

Hence the result.

12. **Let L : V → W be a linear transformation of finitely generated vector spaces V and W. Prove that L is completely determined if we know the images of a basis of V under L. (April 2011)**

Sol. (i) To show that $T(v_i) = w_i$

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V

$\therefore$ For $v \in$ V, $\exists$ unique scalars $a_1, a_2, \ldots, a_n$ such that

$v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$

[$\because$ every element of a V.S. can uniquely be written as the linear Combination of the elements of its basis set]

We define T : V → W by

$T(v) = a_1w_1 + a_2w_2 + \ldots + a_nw_n$

Now since $a_i's$ are unique

$\therefore$ T is well defined.

For $v_i \in V, v_i$ can be written as:

$v_i = 0.v_1 + 0.v_2 + \ldots + 1.v_i + \ldots + 0.v_n$

$\Rightarrow T(v_i) = 0.w_1 + 0.w_2 + \ldots + 1.w_i + \ldots + 0.w_n$

$\Rightarrow T(v_i) = w_i$

(ii) To show that T is linear

Let $x, y \in V$, then $\exists$ scalars $\alpha_i's, \beta_i's\ (1 \le i < n)$

Such that $x = \alpha_1v_1 + \alpha_2v_2 + \ldots + \alpha_nv_n$

and $y = \beta_1v_1 + \beta_2v_2 + \ldots + \beta_nv_n$

$\Rightarrow T(x) = \alpha_1w_1 + \alpha_2w_2 + \ldots + \alpha_nw_n$

and $T(y) = \beta_1w_1 + \beta_2w_2 + \ldots + \beta_nw_n$

Now for $\alpha, \beta \in F$

$\alpha x + \beta y = \alpha(\alpha_1v_1 + \alpha_2v_2 + \ldots + \alpha_nv_n) + \beta(\beta_1v_1 + \beta_2v_2 + \ldots + \beta_nv_n)$

$= (\alpha\alpha_1 + \beta\beta_1)v_1 + (\alpha\alpha_2 + \beta\beta_2)v_2 + \ldots + (\alpha\alpha_n + \beta\beta_n)v_n$

$\therefore T(\alpha x + \beta y) = (\alpha\alpha_1 + \beta\beta_1)w_1 + (\alpha\alpha_2 + \beta\beta_2)w_2 + \ldots + (\alpha\alpha_n + \beta\beta_n)w_n$

$= (\alpha\alpha_1w_1 + \alpha\alpha_2w_2 + \ldots + \alpha\alpha_nw_n) + (\beta\beta_1w_1 + \beta\beta_2w_2 + \ldots + \beta\beta_nw_n)$

$= \alpha(a_1w_1 + \alpha_2w_2 + \ldots + \alpha_nw_n) + \beta(\beta_1w_1 + \beta_2w_2 + \ldots + \beta_nw_n)$

$= \alpha T(x) + \beta T(y)$

$\Rightarrow$ T is linear

(iii) To show T is unique.

Let, if possible, $T' : V \to W$ be another L.T. such that

$T'(v_i) = w_i; 1 \le i \le n$

Let v be an arbitrary element of V, then v can uniquely be expressed as

$v = b_1v_1 + b_2v_2 + \ldots + b_nv_n, \quad b_i's \in F$

$\therefore\ T(v) = b_1w_1 + b_2w_2 + \ldots + b_nw_n$

and $T'(v) = T'(b_1v_1 + b_2v_2 + \ldots + b_nv_n)$

$= b_1T'(v_1) + b_2T'(v_2) + \ldots + b_nT'(v_n)$

$= b_1w_1 + b_2w_2 + \ldots + b_nw_n$

$= T(v)$

i.e. $T'(v) = T(v)\ \forall v \in$ V

$\Rightarrow$ T′ = T $\Rightarrow$ T is unique

Hence the proof.

13. **Let T : V(F) → W(F) be a linear transformation. Prove that image of a linearly dependent set is linearly dependent and pre image of linearly independent set is linearly independent. (September 2010)**

Sol. Since $v_1, v_2, \ldots, v_n \in V$ are L.D. over F

$\therefore \exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ (not all zero)

Such that $\alpha_1v_1 + \alpha_2v_2 + \ldots \alpha_nv_n = 0$

$\Rightarrow$ $T(\alpha_1v_1 + \alpha_2v_2 + \ldots \alpha_nv_n) = T(0)$

$\Rightarrow \alpha_1T(v_1) + \alpha_2T(v_2) + \ldots + \alpha_nT(v_n) = 0$

[$\because$ T is a L.T. and T(0) = 0]

$\Rightarrow T(v_1), T(v_2), ..., T(v_n) \in W$ are L.D.

[$\because$ all $\alpha_i's, 1 \le i \le n$, are not zero]

Second Part:

Let there exists scalars $\beta_1, \beta_2, ..., \beta_n \in F$

Such that $\beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n = 0$ (1)

$\Rightarrow T(\beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n) = T(0)$ [$\because$ T is a L.T.]

$\Rightarrow \beta_1 T(v_1) + \beta_2 T(v_2) + ... + \beta_n T(v_n) = 0$

$\Rightarrow \beta_1 = \beta_2 = ... = \beta_n = 0$ [$\because T(v_1), T(v_2), ..., T(v_n)$ are L.I.]

$\therefore$ From (1), $v_1, v_2, ..., v_n$ are L.I. Hence the proof.

14. **$T: R^3 \to R^3$ be defined by $T(x,y,z) = (x+2y, y-z, x+2z)$**
Verify that Rank T + Nullity T = dim V. **(September 2010)**

Sol. **(i) Range space of T.**

We know that basis of R^3 is

$B = \{(1,0,0),(0,1,0),(0,0,1)\}$

We have $T(1,0,0) = (1+0, 0-0, 1+0) = (1,0,1)$

$T(0,1,0) = (0+2, 1-0, 0+0) = (2,1,0)$

$T(0,0,1) = (0+0, 0-1, 0+2) = (0,-1,2)$

The vectors T(1, 0, 0), T(0, 1, 0), T(0, 0, 1) generate range T
i.e. the vectors (1, 0, 1), (2, 1, 0), (0, –1, 2) generate range T.
To find the basis for range T, we have to determine L.I. vectors from (1, 0, 1), (2, 1, 0), (0, –1, 2).
Consider a matrix A whose rows are the generators of range T.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad [\text{By } R_2 \to R_2 - 2R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{By } R_3 \to R_3 + R_2]$$

$\therefore B_1 = \{(1,0,1),(0,1,-2)\}$ is a basis for R(T)

$\Rightarrow$ Rank (T) = dim (R(T)) = 2

(ii) Null space of T.

Let v = (x, y, z) $\in$ N(T)

$\Rightarrow T(x,y,z) = 0$

$\Rightarrow (x+2y, y-z, x+2z) = (0,0,0)$

$\Rightarrow x+2y = 0,\ y-z = 0,\ x+2z = 0$ (1)

Consider the co-efficient matrix

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \quad [\text{By } R_3 \to R_3 - R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{By } R_3 \to R_3 - 2R_2]$$

The system of equation (1) is equivalent to
x + 2y = 0
y – z = 0
$\Rightarrow$ y = z, x = –2z

$\therefore$ the solution set is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} z$

$\Rightarrow B_2 = \{(-2,1,1)\}$ is a basis for N(T)

$\Rightarrow$ Nullity T = dim (N(T)) = 1
Now Rank T + Nullity T = 2 + 1 = 3 = dim R^3
Hence the verification of Rank-Nullity theorem.

15. **Prove that every n-dimensional vector space over the field F is isomorphic to the space F^n.** **(April 2010)**

Sol. Let V be an n-dimensional vector space over the field F.

Let $B_1 = \{v_1, v_2, ..., v_n\}$ be an ordered basis of V.

We define a mapping $T: V \to F^n$ as:

If $x \in V$ and $x = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n, \alpha_i's \in F$

Then $T(x) = (\alpha_1, \alpha_2, ..., \alpha_n) \in F^n$

To prove that $V \cong F^n$ we have to prove that T is linear, one-one and onto.

T is linear.

Let $x, y \in V$ and $\alpha, \beta \in F$

Now $x = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$

and $y = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$ $\quad \alpha_i's, \beta_i's \in F$

So that $T(x) = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $T(y) = (\beta_1, \beta_2, ..., \beta_n)$

Now $\alpha x + \beta y = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n) + \beta(\beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n)$

$= (\alpha\alpha_1 + \beta\beta_1)v_1 + (\alpha\alpha_2 + \beta\beta_2)v_2 + ... + (\alpha\alpha_n + \beta\beta_n)v_n$

$\Rightarrow T(\alpha x + \beta y) = (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2, ..., \alpha\alpha_n + \beta\beta_n)$

[by def. of T]

$= \alpha(\alpha_1, \alpha_2, ..., \alpha_n) + \beta(\beta_1, \beta_2, ..., \beta_n)$

$= \alpha T(x) + \beta T(y)$

$\therefore$ T is linear.

T is one-one.

For x, y $\in$ V,

$T(x) = T(y)$

$\Rightarrow (\alpha_1, \alpha_2, ..., \alpha_n) = (\beta_1, \beta_2, ..., \beta_n)$

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n$

$\Rightarrow x = y$

$\therefore$ T is one-one

T is onto.

For $(\alpha_1, \alpha_2, ..., \alpha_n) \in F^n$

and $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = x \in V$

we have $T(x) = (\alpha_1, \alpha_2, ..., \alpha_n)$

$\therefore$ T is onto

Since $T: V \to F^n$ is a L.T. which is one-one and onto

$\therefore V \cong F^n$

16. Let $T: V_3(R) \to V_3(R)$ be defined as $T(x,y,z) = (3x, x-y, 2x+y+z)$.
Prove T is invertible and find T^{-1}. **(April 2009)**

Sol. We know that T is invertible iff T is one-one and onto.

(i) To prove T is one-one.

Let $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in V_3(R)$

Such that $T(v_1) = T(v_2)$

$\Rightarrow T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$

$\Rightarrow (3x_1, x_1 - y_1, 2x_1 + y_1 + z_1) = (3x_2, x_2 - y_2, 2x_2 + y_2 + z_2)$

$\Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$

$x_1 - y_1 = x_2 - y_2 \Rightarrow y_1 = y_2$ $\quad [\because x_1 = x_2]$

and $2x_1 + y_1 + z_1 = 2x_2 + y_2 + z_2 \Rightarrow z_1 = z_2$ $\quad [\because x_1 = x_2$ and $y_1 = y_2]$

$\therefore (x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\Rightarrow v_1 = v_2$

Thus $T(v_1) = T(v_2) \quad \Rightarrow v_1 = v_2$

$\therefore$ T is one-one.

(iii) To prove T is onto. Let $(a,b,c) \in V_3(R)$ and we shall show that there exists a vector $(x,y,z) \in V_3(R)$ such that

$T(x,y,z) = (a,b,c)$

$\Rightarrow (3x, x-y, 2x+y+z) = (a,b,c)$

$\Rightarrow 3x = a, x-y = b, 2x+y+z = c$

$\Rightarrow x = \frac{a}{3}, y = \frac{a}{3} - b, z = c - a + b$ $\quad$ [On simplification]

Since $a,b,c \in R \Rightarrow x,y,z \in R$

$\therefore (x,y,z) = \left(\frac{a}{3}, \frac{a}{3} - b, c - a + b\right) \in V_3(R)$

Thus T is onto.

Hence T is one-one and onto

$\Rightarrow$ T is invertible

$\therefore T(x,y,z) = (a,b,c)$

$\Rightarrow T^{-1}(a,b,c) = (x,y,z)$

$= \left(\frac{a}{3}, \frac{a}{3} - b, c - a + b\right)$

$\Rightarrow T^{-1}(a,b,c) = \left(\frac{a}{3}, \frac{a}{3} - b, -a + b + c\right)$ is the required inverse of T.

5

MATRICES AND LINEAR TRANSFORMATIONS

1. **Let $v = R^3$, Find the matrix of standard basis (e_1, e_2, e_3) relative to (f_1, f_2, f_3) where $f_1 = (1, \cos x, \sin x), f_2 = (1,0,0), f_3 = (1, -\sin x, \cos x)$.** **(September 2013)**

Sol. Let $(a,b,c) \in R^3$ and

$(a,b,c) = \alpha(1, \cos x, \sin x) + \beta(1,0,0) - \gamma(1, -\sin x, \cos x)$

$\Rightarrow a = \alpha + \beta + \gamma,\ b = \alpha \cos x - \gamma \sin x,\ c = \alpha \sin x + \gamma \cos x$

Solving

$\alpha = b\cos x + c\sin x$

$\beta = a - b(\cos x - \sin x) - c(\sin x + \cos x)$

$\gamma = -b\sin x + c\cos x$

$\therefore (a,b,c) = (b\cos x + c\sin x) f_1$

$+ (a - b(\cos x - \sin x) - c(\sin x + \cos x)) f_2$

$+ (-b\sin x + c\cos x) f_3$

$\Rightarrow e_1 = (1,0,0) = 0 f_1 + 1 f_2 + 0 f_3.$

$e_2 = (0,1,0) = (\cos x) f_1 - (\cos x - \sin x) f_2 - (\sin x) f_3$

$e_3 = (0,0,1) = (\sin x) f_1 - (\sin x + \cos x) f_2 + (\cos x) f_3$

$$\Rightarrow \text{coefficient matrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos x & \sin x - \cos x & -\sin x \\ \sin x & -\sin x - \cos x & \cos x \end{bmatrix}$$

$\therefore$ B = Matrix of (e_1, e_2, e_3) relative to (f_1, f_2, f_3)

$$= \begin{bmatrix} 0 & \cos x & \sin x \\ 1 & \sin x - \cos x & -\sin x - \cos x \\ 0 & -\sin x & \cos x \end{bmatrix}$$

2. **Let $V = R^3$. Find the matrix of standard basis (e_1, e_2, e_3) relative to (f_1, f_2, f_3), where $f_1 = (2,1,0), f_2 = (0,2,1), f_3 = (0,1,2)$.** **(September 2012)**

Sol. Let $(x,y,z) \in R^3$ and $(x,y,z) = \alpha(2,1,0) + \beta(0,2,1) + \gamma(0,1,2)$ for scalar α, β, γ

$\Rightarrow x = 2\alpha, y = \alpha + 2\beta + \gamma, z = \beta + 2\gamma$

Solving $\alpha = \frac{x}{2}, \beta = \frac{-x+2y-z}{3}, \gamma = \frac{x-2y+4z}{6}$

$\therefore (x,y,z) = \frac{x}{2}(2,1,0) + \frac{-x+2y-z}{3}(0,2,1) + \frac{x-2y+4z}{6}(0,1,2)$

Now $e_1 = (1,0,0) = \frac{1}{2}(2,1,0) + \frac{-1+0-0}{3}(0,2,1) + \frac{1-0+0}{6}(0,1,2)$

$= \frac{1}{2} f_1 - \frac{1}{3} f_2 + \frac{1}{6} f_3$

$e_2 = (0,1,0) = \frac{0}{2}(2,1,0) + \frac{-0+2-0}{3}(0,2,1) + \frac{0-2+0}{6}(0,1,2)$

$= 0 f_1 + \frac{2}{3} f_2 - \frac{1}{3} f_3$

$e_3 = (0,0,1) = \frac{0}{2}(2,1,0) + \frac{-0+0-1}{3}(0,2,1) + \frac{0-0+4}{6}(0,1,2)$

$= f_1 - \frac{1}{3} f_2 + \frac{2}{3} f_3$

$$\Rightarrow \text{Coefficient Matrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\therefore \text{B} = \text{Matrix of } (e_1, e_2, e_3) \text{ w.r.t } (f_1, f_2, f_3) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}^t$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

3. Let $V = R^3$, and let T : V → V be the linear transformation defined by $T(x,y,z) = (2x,4y,5z)$. Find the matrix of T with respect to the basis $\left(\frac{2}{3},0,0\right)$, $\left(0,\frac{1}{2},0\right)$, $\left(0,0,\frac{1}{4}\right)$ of V. **(April 2012)**

Sol. Expressing v = (a, b, c) ∈ R^3 as linear combination of elements of basis (a, b, c) =

$$p\left(\frac{2}{3},0,0\right)+q\left(0,\frac{1}{2},0\right)+r\left(0,0,\frac{1}{4}\right)$$

$$\Rightarrow (a,b,c)=\left(\frac{2}{3}p,\frac{1}{2}q,\frac{1}{4}r\right)$$

$$\Rightarrow a=\frac{2}{3}p,\ b=\frac{1}{2}q,\ c=\frac{1}{4}r$$

$$\Rightarrow p=\frac{3}{2}a,\ q=2b,\ r=4c$$

Now $T: R^3 \to R^3$ is given by

$T(x,y,z)=(2x,4y,5z)$

$$\therefore T\left(\frac{2}{3},0,0\right)=\left(\frac{4}{3},0,0\right)=2\left(\frac{2}{3},0,0\right)+0\left(0,\frac{1}{2},0\right)+0\left(0,0,\frac{1}{4}\right)$$

$$T\left(0,\frac{1}{2},0\right)=(0,2,0)=0\left(\frac{2}{3},0,0\right)+4\left(0,\frac{1}{2},0\right)+0\left(0,0,\frac{1}{4}\right)$$

$$T\left(0,0,\frac{1}{4}\right)=\left(0,0,\frac{5}{4}\right)=0\left(\frac{2}{3},0,0\right)+0\left(0,\frac{1}{2},0\right)+5\left(0,0,\frac{1}{4}\right)$$

$$\therefore [T:B]=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}^t=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

4. Let T be a linear operator on R_3 defined by $T(x,y,z)=(2y+z,x-4y,3x)$. Find the matrix of relative to the Basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$ and verify that [T ; B] [v ; B] = [T(v); B]. **(September 2011)**

Sol. (i) Firstly, we shall express any element

$v=(\alpha,\beta,\gamma)\in R^3$ as a liner combination of the elements of basis B

Let $(\alpha,\beta,\gamma)=a(1,1,1)+b(1,1,0)+c(1,0,0)$ for some reals a, b, c

$=(a+b+c,a+b,a)$

$\Rightarrow a+b+c=\alpha,\quad a+b=\beta,\quad a=\gamma$

Solving these, we get, $a=\gamma,\quad b=\beta-\gamma,\quad c=\alpha-\beta$

$\therefore (\alpha,\beta,\gamma)=\gamma(1,1,1)+(\beta-\gamma)(1,1,0)+(\alpha-\beta)(1,0,0)$ (1)

Given T : $R^3 \to R^3$ is a linear operator defined as

$T(x,y,z)=(2y+z,x-4y,3x)$

and $B=\{(1,1,1),(1,1,0),(1,0,0)\}$ is a basis of R^3

Now $T(1,1,1)=(2+1,1-4,3)=(3,-3,3)$

$=3(1,1,1)+(-3-3)(1,1,0)+(3+3)(1,0,0)$ [Using (1)]

$=3(1,1,1)+(-6)(1,1,0)+(6)(1,0,0)$

$T(1,1,0)=(2+0,1-4,3)=(2,-3,3)$

$=3(1,1,1)+(-3-3)(1,1,0)+(2+3)(1,0,0)$ [Using (1)]

$=3(1,1,1)+(-6)(1,1,0)+(5)(1,0,0)$

$T(1,1,0)=(0+0,1-0,3)=(0,1,3)$

$=3(1,1,1)+(1-3)(1,1,0)+(0-3)(1,0,0)$ [Using (1)]

$=3(1,1,1)+(-2)(1,1,0)+(-1)(1,0,0)$

$$\therefore [T:B]=\begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^t=\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

(ii) To verify that [T ; B] [v ; B] = [T(v) ; B] ∀ $v\in R^3$.

Let $v=(x,y,z)\in R^3$

Then $v=(x,y,z)=z(1,1,1)+(y-z)(1,1,0)+(x-y)(1,0,0)$ [Using (1)]

$$\therefore [v\,;B]=[z,y-z,x-y]'=\begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

Now $T(v)=T(x,y,z)$

$=(2y+z,x-4y,3x)$ [by def. of T]

$=3x(1,1,1)+(x-4y-3x)(1,1,0)+(2y+z-x+4y)(1,0,0)$ [Using (1)]

$=3x(1,1,1)+(-2x-4y)(1,1,0)+(-x+6y+z)(1,0,0)$

$$\therefore [T(v)\,;B]=[3x \quad -2x-4y \quad -x+6y+z]'=\begin{bmatrix} 3x \\ -2x-4y \\ -x+6y+z \end{bmatrix}$$

L.H.S. = [T ; B] [v ; B]

$$=\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}\begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

$$=\begin{bmatrix} 3z+3(y-z)+3(x-y) \\ -6z-6(y-z)-2(x-y) \\ 6z+5(y-z)-1(x-y) \end{bmatrix}$$

$$=\begin{bmatrix} 3z+3y-3z+3x-3y \\ -6z-6y+6z-2x+2y \\ 6z+5y-5z-x+y \end{bmatrix}=\begin{bmatrix} 3x \\ -2x-4y \\ -x+6y+z \end{bmatrix}$$

= [T(v) ; B]

= R.H.S.

Hence the result is verified.

5. **Let $S=\{(1,2),(0,1)\}$ and $T=\{(1,1),(2,3)\}$ be bases for R^2. What is the transition matrix from T to S basis?** **(April 2011)**

Sol. Let $A=\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}'$ is transition matrix from S to T.

$\Rightarrow (1,1)=a_{11}(1,2)+a_{21}(0,1)$

$(2,3)=a_{12}(1,2)+a_{22}(0,1)$

$\Rightarrow (1,1)=(a_{11},2a_{11}+a_{21})$

$\Rightarrow a_{11}=1,\ 2a_{11}+a_{21}=1\Rightarrow a_{21}=-1$

Also $(2,3)=(a_{12},2a_{12}+a_{22})$

$\Rightarrow a_{12}=2,\ 2a_{12}+a_{22}=3\Rightarrow a_{22}=-1$

Hence transition matrix $=\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

6. **Let T be a linear operator on R^2 defined by: $T(x,y)=(4x-2y,2x+y)$. Find the matrix of T relative to the basis $B=\{(1,1);(-1,0)\}$.** **(April 2010)**

Also verify that [T ; B] [v ; B] = [T(v) ; B] for any vector $v\in R^2$.

Sol. Firstly, we shall express any element

$v=(\alpha,\beta)\in R^2$ as a linear combination of the element of basis B.

Let $(\alpha,\beta)=a(1,1)+b(-1,0)$ for reals a and b

$\Rightarrow (\alpha,\beta)=(a-b,a)$

$\therefore \alpha=a-b,\ \beta=a$

$\Rightarrow a=\beta$ and $b=\beta-\alpha$

$\therefore (\alpha,\beta)=\beta(1,1)+(\beta-\alpha)(-1,0)$ —— (1)

Given $T:R^2\to R^2$ defined as

$T(x,y)=(4x-2y,2x+y)$

and $B=\{(1,1),(-1,0)\}$ is a basis of R^2

Now $T(1,1)=(4-2,2+1)=(2,3)=3(1,1)+(3-2)(-1,0)$ [Using (1)]

$=3(1,1)+1(-1,0)$

and $T(-1,0)=(-4-0,2(-1)+0)$

$=(-4,-2)=(-2)(1,1)+(-2+4)(-1,0)$ [Using (1)]

$=-2(1,1)+2(-1,0)$

$$\therefore [T:B]=\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}'=\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

Which is the matrix of T relative to the basis B.

To verify $[T;B][v;B]=[T(v);B]$

Let $v=(x,y)\in R^2$

Then $v=(x,y)=y(1,1)+(y-x)(-1,0)$ [Using (1)]

$$\therefore [v;B]=[y \quad y-x]'=\begin{bmatrix} y \\ y-x \end{bmatrix}$$

Now $T(v)=T(x,y)$

$=(4x-2y,2x+y)$ (by def. of T)

$= (2x+y)(1,1) + (2x+y-4x+2y)(-1,0)$ [Using (1)]

$= (2x+y)(1,1) + (-2x+3y)(-1,0)$

$\therefore \left[T(v);B\right] = \left[2x+y \quad -2x+3y\right]^t = \begin{bmatrix} 2x+y \\ -2x+3y \end{bmatrix}$

L.H.S. = [T ; B] [v ; B]

$= \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y \\ y-x \end{bmatrix}$

$= \begin{bmatrix} 3y-2(y-x) \\ y+2(y-x) \end{bmatrix}$

$= \begin{bmatrix} 2x+y \\ -2x+3y \end{bmatrix} = \left[T(v);B\right] = R.H.S.$

Hence the result is verified.

7. Find matrix representation of the linear transformation $T : R^2 \to R^3$ defined by $T(x,y) = (3x-2y, 0, x+4y)$ with respect to basis $B_1 = \{(1,1),(0,2)\}$ and $B_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$ for R^2 and R^3 respectively. (September 2009)

Sol. Given $T : R^2 \to R^3$ defined by

$T(x,y) = (3x-2y, 0, x+4y)$

and $B_1 = \{(1,1),(0,2)\}$

$B_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$ be ordered basis for R^2 and R^3 respectively.

To find [T ; B_1, B_2]

Firstly we shall express any vector $v = (\alpha,\beta,\gamma) \in R^3$ as a linear combination of the elements of basis B_2.

Let $(\alpha,\beta,\gamma) = a(1,1,0) + b(1,0,1) + c(0,1,1)$ for some scalars a, b, c

$= (a+b, a+c, b+c)$

$\Rightarrow a+b = \alpha, a+c = \beta, b+c = \gamma$

On solving these, we get

$a = \frac{\alpha+\beta-\gamma}{2}, b = \frac{\alpha-\beta+\gamma}{2}, c = \frac{-\alpha+\beta+\gamma}{2}$

$\therefore (\alpha,\beta,\gamma) = \frac{\alpha+\beta-\gamma}{2}(1,1,0) + \frac{\alpha-\beta+\gamma}{2}(1,0,1) + \frac{-\alpha+\beta+\gamma}{2}(0,1,1)$ (1)

Now T(1, 1) = (3 – 2, 0, 1 + 4) [by def of T]

= (1, 0, 5)

$= \frac{1+0-5}{2}(1,1,0) + \frac{1-0+5}{2}(1,0,1) + \frac{-1+0+5}{2}(0,1,1)$

$= -2(1,1,0) + 3(1,0,1) + 2(0,1,1)$

and T(0, 2) = (0 – 4, 0, 0 + 8) = (–4, 0, 8) [by def of T]

$= \frac{-4+0-8}{2}(1,1,0) + \frac{-4-0+8}{2}(1,0,1) + \frac{4+0+8}{2}(0,1,1)$

$= -6(1,1,0) + 2(1,0,1) + 6(0,1,1)$

$\therefore [T ; B_1, B_2] = \begin{bmatrix} -2 & 3 & 2 \\ -6 & 2 & 6 \end{bmatrix}^t = \begin{bmatrix} -2 & -6 \\ 3 & 2 \\ 2 & 6 \end{bmatrix}$

8. Let T : V → V be a linear operator, where V is a finite dimensional vector space over F (a field). Suppose $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V(F). Prove that for any vector v ∈ V [T ; B] [v ; B] = [T(v) ; B]. (April 2009)

Sol. Given T : V → V be a L.T.

and B = $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

Let $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i, 1 \le j \le n$

$\therefore [T ; B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$ [By definition]

Let v be any element of V

$\therefore$ for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

We have, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

i.e. $v = \sum_{j=1}^{n} \alpha_j v_j$

then $[v ; B] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \cdots \\ \alpha_j \\ \cdots \\ \cdots \\ \alpha_n \end{bmatrix}$

Now since $v = \sum_{j=1}^{n} \alpha_j v_j$

$$\therefore T(v) = T\left(\sum_{j=1}^{n} \alpha_j v_j\right)$$

$$= \sum_{j=1}^{n} \alpha_j T(v_j) \qquad [\because \text{T is a L.T.}]$$

$$= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{n} a_{ij} v_i\right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}\alpha_j\right) v_i$$

$$= \sum_{i=1}^{n} \left(a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots\ldots + a_{ij}\alpha_j + \ldots\ldots\ldots + a_{in}\alpha_n\right) v_i$$

$$\Rightarrow [T(v)\,;\,B] = \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots\ldots\ldots + a_{ij}\alpha_j + \ldots\ldots\ldots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots\ldots\ldots + a_{2j}\alpha_j + \ldots\ldots\ldots + a_{2n}\alpha_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots\ldots\ldots + a_{nj}\alpha_j + \ldots\ldots\ldots + a_{nn}\alpha_n \end{bmatrix}$$

Also

$$[T\,;\,B]\,[v\,;\,B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_j \\ \ldots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots\ldots\ldots + a_{1j}\alpha_j + \ldots\ldots\ldots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots\ldots\ldots + a_{2j}\alpha_j + \ldots\ldots\ldots + a_{2n}\alpha_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots\ldots\ldots + a_{nj}\alpha_j + \ldots\ldots\ldots + a_{nn}\alpha_n \end{bmatrix}$$

Hence $[T\,;\,B]\,[v\,;\,B] = [T(v)\,;\,B]$

6

CHARACTERISTIC ROOTS & CHARACTERISTIC VECTORS OF A MATRIX

Prabhjot Kaur

1. **Prove that non zero eigen vectors belonging to distinct eigen values are linearly independent.**

(September 2013, April 2012, September 2011, September 2010)

Sol. Let $X_1, \ldots\ldots, \ldots, X_m$ be the characteristic vectors of a matrix A corresponding to distinct characteristic values $\lambda_1, \ldots\ldots, \lambda_m$. To prove that the vectors $X_1, \ldots\ldots, X_m$ are linearly independent.

If the vector $X_1, \ldots\ldots, X_m$ are linearly dependent we can choose r so that $1 \le r < m$ and $X_1, \ldots\ldots, X_r$ are linearly independent but $X_1, \ldots\ldots, X_r, X_{r+1}$ are linearly dependent. Hence we can choose scalars $a_1, \ldots\ldots, a_{r+1}$, not all zero such that

$$a_1X_1 + \ldots + a_{r+1}X_{r+1} = O \qquad (1)$$

$$\Rightarrow A(a_1X_1 + \ldots + a_{r+1}X_{r+1}) = AO$$

$$\Rightarrow a_1AX_1 + \ldots + a_{r+1}AX_{r+1} = O$$

$$\Rightarrow a_1(\lambda_1X_1) + \ldots + a_{r+1}(\lambda_{r+1}X_{r+1}) = O \qquad (2)$$

Multiplying (1) be the scalar λ_{r+1} and subtracting from (2), we get

$$a_1(\lambda_1 - \lambda_{r+1})X_1 + \ldots + a_r(\lambda_r - \lambda_{r+1})X_r = O \qquad (3)$$

Since $X_1, \ldots\ldots, X_r$ are linearly independent according to our assumption and $\lambda_1, \ldots\ldots, \lambda_{r+1}$ are distinct, therefore from (3), we get

$$a_1 = 0, \ldots\ldots, a_r = 0$$

Putting $a_1 = 0, \ldots\ldots, a_r = 0$ in (1), we get

$$a_{r+1}X_{r+1} = O$$

$\Rightarrow a_{r+1} = 0$ since $X_{r+1} \ne O$.

Thus the relation (1) implies that

$a_1 = 0, \ldots\ldots, a_r = 0, a_{r+1} = 0$

But this contradicts our assumption that the scalars

$a_1, \ldots\ldots, a_{r+1}$ are not all zero.

Hence our initial assumption is wrong and the vectors $X_1, \ldots\ldots, X_m$ are linearly independent.

2. Find all the eigen values and eigen vectors of the matrix: **(September 2013)**

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & 6 & 4 \end{bmatrix}$$

Sol. Given $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & 6 & 4 \end{bmatrix}$

The eigen values of A are the values of t s.t. $|tI - A| = 0$

$$\begin{vmatrix} (t-1) & 3 & -3 \\ -3 & (t+5) & -3 \\ -6 & -6 & (t-4) \end{vmatrix} = 0$$

Operate, $C_1 \to C_1 + C_2,\ C_2 \to C_2 + C_3$

$$\Rightarrow \begin{vmatrix} (t+2) & 0 & -3 \\ -3 & (t+2) & -3 \\ 0 & (t+2) & (t-4) \end{vmatrix} = 0$$

$$\Rightarrow (t+2)^2 \begin{vmatrix} 1 & 0 & -3 \\ 1 & 1 & -3 \\ 0 & 1 & (t-4) \end{vmatrix} = 0$$

Operate, $R_2 \to R_2 - R_1$

$$\Rightarrow (t+2)^2 \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 1 & (t-4) \end{vmatrix} = 0 \qquad \text{Expand by } C_1$$

$$\Rightarrow (t+2)^2 \begin{vmatrix} 1 & 0 \\ 1 & (t-4) \end{vmatrix} = 0$$

or $(t+2)^2 (t-4) = 0$

$\Rightarrow t = -2, -2, 4$

$\Rightarrow t = -2, 4$ are two distinct eigen values of A.

To find eigen vectors of A associated to eigen value –2.

Putting, $t = -2$ in $(tI - A)X = O$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value –2.

$$\therefore \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate, $R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} -3 & 3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ We get, $-3x + 3y - 3z = 0$ or $-x + y - z = 0$ or $x = y - z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

Thus, $X_1 = (1, 1, 0)$ and $X_2 = (1, 0, 1)$ are two L.I. given vectors of A associated to eigen value –2.

To find eigen vector of A associated to eigen value 4:

Putting, $t = 4$ in $(tI - A)X = O$

where, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value 4.

$$\therefore \begin{bmatrix} -3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate, $R_2 \to R_2 + R_1,\ R_3 \to R_3 + 2R_2$

$$\Rightarrow \begin{bmatrix} 3 & 3 & -3 \\ 0 & 12 & -6 \\ 0 & 12 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 3 & 3 & -3 \\ 0 & 12 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ we get,

$3x+3y-3z=0$ or $x+y-z=0$

and $12y-6z=0 \Rightarrow 2y-z=0$

or $\left.\begin{matrix} x+y=z \\ 2y=z \end{matrix}\right] \Rightarrow \begin{matrix} x=y \\ z=2y \end{matrix}$

$\therefore$ y is free variable.

$$\therefore x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y$$

$\Rightarrow$ (1, 1, 2) is the eigen vector associated to eigen value $t = 4$.

3. **Use Cayley's Hamilton theorem to find inverse of** **(April 2013)**

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Sol. Characteristic Equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \left|\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(1-\lambda)(3-\lambda) = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

According to Cayley's – Hamilton theorem, every square matrix satisfies its characteristic equation

$\Rightarrow A^3 - 5A^2 - 7A - 3I = 0$

$\Rightarrow A^{-1}(A^3 - 5A^2 + 7A - 3I) = 0$

$\Rightarrow A^2 - 5A + 7I - 3A^{-1} = 0$

$\Rightarrow A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$

$$\Rightarrow A^{-1} = \frac{1}{3}\left[\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -5 \\ 0 & 5 & 10 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix} 3 & -6 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

4. **Prove that eigen values of Skew Hermitian matrix are purely imaginary.** **(April 2013, April 2011)**

Sol. Let A be a skew Hermitian matrix

Then $A^\theta = -A$.

Now $(iA)^\theta = \bar{i}A^\theta$

$= (-i)(-A)$

$= iA$

$\Rightarrow iA$ is a Hermitian matrix

Let λ be a characteristic root of A

$\therefore AX = \lambda X, X \neq O$

$\Rightarrow (iA)X = (i\lambda)X$

$\Rightarrow i\lambda$ is a characteristic root of hermitian matrix iA

$\Rightarrow i\lambda$ is real [$\because$ characteristic roots of a Hermitian matrix are real]

$\Rightarrow$ either λ = 0 or λ is purely imaginary.

5. **Prove that eigen values of a Hermitian matrix are real.** **(September 2012)**

Sol. Let A be a Hermitian matrix and λ be the characteristic root of A.

Let X be the characteristic vector corresponding to characteristic root λ.

$\therefore AX = \lambda X$

Pre-multiplying by X^θ, we get

$X^\theta(AX) = X^\theta(\lambda X)$

$\Rightarrow X^\theta AX = \lambda X^\theta X$ (1)

$\Rightarrow (X^{\theta}AX)^{\theta} = (\lambda X^{\theta}X)^{\theta}$

$\Rightarrow X^{\theta}A^{\theta}(X^{\theta})^{\theta} = \bar{\lambda}X^{\theta}(X^{\theta})^{\theta}$

$\Rightarrow X^{\theta}AX = \bar{\lambda}X^{\theta}X$ **(2)**

$\left[\because A^{\theta} = A \text{ and } (X^{\theta})^{\theta} = X\right]$

From (1) and (2), we get,

$\lambda X^{\theta}X = \bar{\lambda}X^{\theta}X$

$\Rightarrow (\lambda - \bar{\lambda})X^{\theta}X = O$

$\Rightarrow \lambda - \bar{\lambda} = 0$ $\quad \left[\because X \neq O \quad \therefore X^{\theta}X \neq O\right]$

$\Rightarrow \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is real.

Hence the result.

6. **Define eigenvalue of a square matrix. Let A be a square matrix of order n. If λ is an eigenvalue of A, then show using definition of eigenvalue that (i) λ^m is an eigenvalue of A^m ; (ii) if A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.**

(April 2012)

Sol. Let $A = [a_{ij}]_{n\times n}$ be any n-rowed square matrix and λ an indeterminate. The matrix A – λI is called characteristic matrix of A where I is unit matrix of order n.

The determinant $|A - \lambda I|$ is characteristic polynomial of A.

and the roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic roots or eigen values

(i) Suppose λ is any characteristic root of A and X is characteristic vector of A

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(\lambda X)$

$\Rightarrow A^2X = \lambda^2X$

Similarly if m is any positive integer, then repeating the above process m times, we get

$A^mX = \lambda^mX$

$\Rightarrow \lambda^m$ is an eigen value of A^m

(ii) Let λ be an eigen value of A and X be corresponding eigen vector. Then

$AX = \lambda X \Rightarrow A^{-1}AX = A^{-1}(\lambda X)$

$\Rightarrow X = \lambda(A^{-1}X)$

$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}

7. **Find the matrix of eigenvectors of the matrix given by $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.**

Is this matrix diagonalizable? Give reasons. **(April 2012)**

Sol. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The eigen values of A are the values of t s.t. $|tI - A| = 0$

$\Rightarrow \begin{vmatrix} t & -1 & -1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} = 0$

$\Rightarrow (t-2)(t+1)^2 = 0$

$\Rightarrow t = 2, -1, -1$ are the eigen values of A

To find eigen vector of A associated to eigen value = 2

Putting t = 2 in (t I – A)X = 0

Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value 2

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 + \frac{1}{2}R_1,\ R_3 \to R_3 + \frac{1}{2}R_1$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & \frac{-3}{2} \\ 0 & \frac{-3}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 + R_2,\ R_2 \to \frac{2}{3}R_2$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x - y - z = 0 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}z = z$

Also $y - z = 0 \Rightarrow y = z$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} z$$

$\therefore$ (1, 1, 1) is eigen vector associated to $\lambda = 2$.

To find eigen vector of A associated to eigen value $\lambda = -1$

Putting $t = -1$ in $(tI - A)X = 0$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value $\lambda = -1$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + y + z = 0 \Rightarrow x = -y - z$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore$ (–1, 1, 0) and (–1, 0, 1) are eigen values corresponding to $\lambda = -1$

Since, A has three L.I. eigen vectors namely (1, 1, 1), (–1, 1, 0) and (–1, 0, 1)

$\therefore$ A is diagonalizable.

8. **Find all the eigen values and a basis for the corresponding eigen space of A.**

$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$ **Is A diagonalizable?** **(September 2011)**

Sol. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

The eigen values of A are the value of t such that

$|tI - A| = 0$

$$\Rightarrow \begin{vmatrix} (t-1) & -2 & -2 \\ -1 & (t-2) & 1 \\ 1 & -1 & (t-4) \end{vmatrix} = 0 \qquad \text{Operate } C_1 \to C_1 + C_2$$

$$\Rightarrow \begin{vmatrix} (t-3) & -2 & -2 \\ (t-3) & (t-2) & 1 \\ 0 & -1 & (t-4) \end{vmatrix} = 0$$

$$\Rightarrow (t-3)\begin{vmatrix} 1 & -2 & -2 \\ 1 & (t-2) & 1 \\ 0 & -1 & (t-4) \end{vmatrix} = 0 \qquad \text{Operate } R_2 \to R_2 - R_1$$

$$\Rightarrow (t-3)\begin{vmatrix} 1 & -2 & -2 \\ 0 & t & 3 \\ 0 & -1 & t-4 \end{vmatrix} = 0$$

$\Rightarrow (t-3)(t^2 - 4t + 3) = 0$ [Expanding by C_1]

$\Rightarrow (t-3)(t-3)(t-1) = 0$

$\Rightarrow t = 3, 3, 1$

$\therefore t = 1, 3$ are two distinct eigen values of A.

To find basis for eigen space of $t = 1$:

Putting $t = 1$ in $|tI - A|X = O$

Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value $t = 1$

$$\Rightarrow \begin{bmatrix} 0 & -2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 0 & -2 & -2 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Operate } R_3 \to R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 0 & -2 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \Rightarrow\ & -2y - 2z = 0 \\ and\ & -x - y + z = 0 \end{aligned}\right] \Rightarrow \begin{aligned} y &= -z \\ y &= -x + z \end{aligned}$$

$\therefore$ we have $y = -z$

And $x = 2z$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} z$$

Hence $\{(2, -1, 1)\}$ is a basis for eigen space of $t = 1$.

To find basis for eigen space of $t = 3$:

Putting $t = 3$ in $(t\,I - A)X = O$

Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the eigen vector of A associated to eigen value $t = 3$

$$\Rightarrow \begin{bmatrix} 2 & -2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 + R_2, R_1 \to R_1 + 2R_2$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Operate } R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x + y + z = 0 \qquad \Rightarrow x = y + z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore \{(1,1,0)(1,0,1)\}$ is a basis of eigen space of $t = 3$.

Since A has three L.I. eigen vectors.

$\therefore$ A is Diagonalizable.

In Fact, let P be the matrix whose columns are three L.I. vectors.

$$\text{i.e., } P = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\text{then } P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence A is Diagonalizable.

9. **Find the eigen values and the eigen-vectors of the matrix** **(April 2011)**

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The eigen value of A are the values of t s.t. $|t\,I - A| = 0$

$$\Rightarrow \begin{vmatrix} t-2 & 0 & -1 \\ 0 & t-2 & 0 \\ -1 & 0 & t-2 \end{vmatrix} = 0$$

$\Rightarrow (t-2)\left[(t-2)^2 - 0\right] - 1\left[0 + (t-2)\right] = 0$

$\Rightarrow (t-2)\left[t^2 - 4t + 3\right] = 0$

$\Rightarrow (t-1)(t-2)(t-3) = 0$

$\Rightarrow$ t = 1, 2, 3 are the eigen values of A

To find eigen vector of A associated to eigen value = 1.

Putting t = 1 in $|t\,I - A|X = 0$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to t = 1

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x = −z
y = 0.z
$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow -x-z=0 \quad \Rightarrow x=-z$

Also $-y=0 \quad \Rightarrow y=0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}z \Rightarrow (-1,0,1)$ is eigen vector

Corresponding to eigen value t = 1

To find eigen vector of A associated to eigen value = 2

Putting t = 2 in $(tI-A)X=0$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ x = 0, z = 0, y can taken any value

$\Rightarrow$ (0, 1, 0) is eigen vector corresponding to eigen value t = 2.

To find eigen vector of A associated to eigen value = 3

Putting t = 3 in $(tI-A)X=0$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x-z=0 \quad \Rightarrow \quad x=z$

Also $\quad y=0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}z \Rightarrow (1,0,1)$ is eigen vector corresponding to eigen value t=3

10. Find all the eigen values and basis of eigenspace of the matrix

$A=\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ **Is A diagonalizable?** **(September 2010)**

Sol. Let $A=\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

Let λ be the characteristic root of A.

Now $[\lambda I-A]=\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}+\begin{bmatrix} 9 & -4 & -4 \\ 8 & -3 & -4 \\ 16 & -8 & -7 \end{bmatrix}$

$=\begin{bmatrix} \lambda+9 & -4 & -4 \\ 8 & \lambda-3 & -4 \\ 16 & -8 & \lambda-7 \end{bmatrix}$

$\therefore |\lambda I-A|=\begin{vmatrix} \lambda+9 & -4 & -4 \\ 8 & \lambda-3 & -4 \\ 16 & -8 & \lambda-7 \end{vmatrix}$

$=\begin{vmatrix} \lambda+1 & -4 & -4 \\ \lambda+1 & \lambda-3 & -4 \\ \lambda+1 & -8 & \lambda-7 \end{vmatrix} \qquad [C_1 \to C_1+C_2+C_2]$

$=(\lambda+1)\begin{vmatrix} 1 & -4 & -4 \\ 1 & \lambda-3 & -4 \\ 1 & -8 & \lambda-7 \end{vmatrix}$

$=(\lambda+1)\begin{vmatrix} 1 & -4 & -4 \\ 0 & \lambda+1 & 0 \\ 0 & -4 & \lambda-3 \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{bmatrix}$

$=(\lambda+1)^2(\lambda-3)$

Now $|\lambda I-A|=0$

$\Rightarrow (\lambda+1)^2(\lambda-3)=0$

$\Rightarrow \lambda=-1,-1,3$

$\therefore$ Characteristic roots of A are −1, −1, 3

To find eigen vectors corresponding to $\lambda=-1$

Let $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the characteristic vector A corresponding to characteristic root −1

$\therefore [-I-A]X=O$

$\Rightarrow \begin{bmatrix} 8 & -4 & -4 \\ 8 & -4 & -4 \\ 16 & -8 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 8 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{bmatrix}$$

$$\Rightarrow 8x - 4y - 4z = 0 \qquad \Rightarrow \qquad z = 2x - y$$

Now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} y$

$\therefore$ (1, 0, 2), (0, 1, –1) are two linearly independent characteristic vectors of A corresponding to characteristic root –1.

To find eigen vectors corresponding to $\lambda = 3$

Let X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the characteristic vector of A corresponding to characteristic root 3.

$\therefore$ [3I – A] X = O

$$\Rightarrow \begin{bmatrix} 12 & -4 & -4 \\ 8 & 0 & -4 \\ 16 & -8 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & -1 \\ 2 & 0 & -1 \\ 0 & -8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left[R_1 \to \frac{1}{4}R_1, \; R_2 \to \frac{1}{4}R_2 \text{ and then } R_3 \to R_3 - 8R_2 \right.$$

$$\Rightarrow 3x - y - z = 0,\; 2x - 1y = 0,\; -8y + 4z = 0$$

$$\Rightarrow z = 2y,\; x = y$$

Now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y$

$\therefore$ (1, 1, 2) is the characteristic vector corresponding to characteristic root 3.
Since the matrix A has three linearly independent characteristic vectors in R^3.
$\therefore$ A is diagonalizable.

11. Find all the eigen values and the eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

(September 2012, April 2010)

Is A diagonalizable?

Sol. Let A = $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

The eigen values of A are the values of t s.t.

$|t\,I - A| = 0$

$$\Rightarrow \begin{vmatrix} (t-3) & -1 & -1 \\ -2 & (t-4) & -2 \\ -1 & -1 & (t-3) \end{vmatrix} = 0 \qquad \text{Operate, } C_1 \to C_1 - C_2$$

$$\Rightarrow \begin{vmatrix} t-2 & -1 & -1 \\ -(t-2) & (t-4) & -2 \\ 0 & -1 & (t-3) \end{vmatrix} = 0$$

$$\text{or, } (t-2)\begin{vmatrix} 1 & -1 & -1 \\ -1 & (t-4) & -2 \\ 0 & -1 & (t-3) \end{vmatrix} = 0 \qquad \text{Expand by } C_1$$

$$(t-2)\left[\left((t-4)(t-3)-2\right)+1(-t+3-1)\right]=0$$

$$\Rightarrow (t-2)\left[t^2-7t+12-2-t+3-1\right]=0$$

$$\Rightarrow (t-2)\left[t^2-8t+12\right]=0$$

$$\text{or } (t-2)(t-2)(t-6)=0$$

$$\Rightarrow t = 2, 2, 6$$

$\therefore$ $t = 2, 6$ are the two distinct eigen values of A.

To find eigen vectors associated to eigen value t = 2

Putting, t = 2 in $(t\,I - A)$ X = 0

where, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value 2.

$$\therefore \begin{bmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate, $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x-y-z=0$ or $x=-y-z$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore$ (–1, 1, 0), (–1, 0, 1) are two L.I. eigen vectors of A associated to eigen value 2.

To find eigen vector of A associated to eigen value 6:

Putting, t = 6 in $(t\,\text{I} - \text{A})\,\text{X} = \text{O}$

Where, $\text{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value 6.

$$\therefore \begin{bmatrix} 3 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate, $R_1 \to R_1 + 3R_3,\ R_2 \to R_2 - 2R_3$

$$\Rightarrow \begin{bmatrix} 0 & -4 & 8 \\ 0 & 4 & -8 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate, $R_1 \leftrightarrow R_3$ and then $R_3 \to R_3 + R_2$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 3 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These can be written as,

$$\left.\begin{matrix} -x-y+3z=0 \\ 4y-8z=0 \end{matrix}\right] \Rightarrow \begin{matrix} x=z \\ y=2z \end{matrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} z$$

$\therefore$ (1, 2, 1) is the L.I. eigen vector of A associated to eigen value 6. Since, A has three L.I. eigen vectors namely, (–1, 1, 0), (–1, 0, 1), (1, 2, 1)

$\therefore$ A is diagonalizable.

12. **Find a unitary matrix that will diagonalize the Hermitian matrix $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$.**

(April 2010)

Sol. Given matrix is $\text{A} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

The characteristic Equation of A is $|\lambda \text{I} - \text{A}| = 0$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -i \\ i & \lambda-1 \end{vmatrix} = 0$$

$\Rightarrow (\lambda-1)^2 + i^2 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 1 - 1 = 0 \qquad \Rightarrow \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda = 0, 2$

$\therefore$ The characteristic roots of A are 0, 2

To find characteristic vectors:

The characteristic vector X corresponding to $\lambda = 0$ is given by (0I – A) X = O

$$\Rightarrow \begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Apply } R_2 \to R_2 + iR_1$$

$$\Rightarrow \begin{bmatrix} -1 & -i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x - iy = 0 \qquad \Rightarrow x + iy = 0$

Clearly x = 1, y = i is a solution

$\therefore \text{X} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is a L.I. characteristic vector corresponding to $\lambda = 0$

The characteristic vector Y corresponding to $\lambda = 2$ is given by (2I – A) Y = O

$$\Rightarrow \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Apply } R_2 \to R_2 - iR_1$$

$$\Rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - iy = 0$

Clearly $x = i, y = 1$

$\therefore Y = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is L.I. characteristic vector corresponding to $\lambda = 2$

Thus characteristic vectors are $X = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $Y = \begin{bmatrix} i \\ 1 \end{bmatrix}$ corresponding to characteristic roots 0, 2

Here X, Y are orthogonal

$$\left(\begin{aligned} \because X^{\theta}Y &= \begin{bmatrix} 1 & \bar{i} \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= (1)(i) + (\bar{i})(1) \\ &= (1)(i) - (i)(1) \\ &= 0 \end{aligned} \right)$$

Now normalize these vectors by dividing components of each vector by its length

Length of $X = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is $\sqrt{1^2 + |i|^2} = \sqrt{1+1} = \sqrt{2}$.

Length of $Y = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is $\sqrt{|i|^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$.

$\therefore$ Normalized vectors are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

then $E = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ which is a transforming Matrix

$\because$ E is unitary $\Rightarrow E^{-1} = E^{\theta}$

$$\text{so } E^{-1}AE = E^{\theta}AE = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -\frac{2i}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

= diag (0, 2)

13. Find characteristic roots and characteristic vector of

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Also verify that geometric multiplicity of a characteristic root can not exceed it's a algebraic multiplicity. **(September 2009)**

Sol. The characteristic root of A are the value of λ such that $|\lambda I - A| = 0$

$$\Rightarrow \begin{vmatrix} \lambda-3 & -10 & -5 \\ 2 & \lambda+3 & 4 \\ -3 & -5 & \lambda-7 \end{vmatrix} = 0$$

Expanding by R_1

$\Rightarrow (\lambda-3)((\lambda+3)(\lambda-7)+20)+10(2(\lambda-7)+12)-5(-10+3(\lambda+3))=0$

$\Rightarrow (\lambda-3)(\lambda^2-4\lambda-1)+10(2\lambda-2)-5(3\lambda-1)=0$

$\Rightarrow \lambda^3-4\lambda^2-\lambda-3\lambda^2+12\lambda+3+20\lambda-20-15\lambda+5=0$ **(i)**

$\Rightarrow \lambda^3-7\lambda^2+16\lambda-12=0$

Clearly $\lambda = 2$ satisfies (By Trial method)

$\therefore$ (i) can be written as $(\lambda-2)(\lambda^2-5\lambda+6)=0$

$\Rightarrow \lambda = 2, 2, 3$

So characteristic roots of A are 2, 2, 3

To find characteristic vector of A associated to $\lambda = 2$

Putting $\lambda = 2$ in $(\lambda I - A)X = O$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is characteristic vector of A associated to characteristic root 2.

$$\Rightarrow \begin{bmatrix} -1 & -10 & -5 \\ 2 & 5 & 4 \\ -3 & -5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 + 2R_1, R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} -1 & -10 & -5 \\ 0 & -15 & -6 \\ 0 & 25 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to -\frac{1}{3}R_2, R_3 \to \frac{1}{5}R_3$

$$\Rightarrow \begin{bmatrix} -1 & -10 & -5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} -1 & -10 & -5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x-10y-5z=0$ and $5y+2z=0$

$\Rightarrow x=-10y-5z$ and $5y=-2z$ or $10y=-4z$

$\Rightarrow x=4z-5z$ and $5y=-2z$

$\Rightarrow x=-z$ and $y=\frac{-2}{5}z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -\frac{2}{5}z \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{2}{5} \\ 1 \end{bmatrix} z$$

$\Rightarrow \left(-1, -\frac{2}{5}, 1\right)$ is L.I. characteristic vector associated to characteristic root $\lambda = 2$.

Since there is one L.I. characteristic vector associated to characteristic root $\lambda = 2$, so that the geometric multiplicity of $\lambda = 2$ is one. Also $\lambda = 2$ is a multiple root of characteristic equation with multiplicity 2

($\because$ 2, 2 are characteristic roots of A)

so that **algebraic multiplicity of $\lambda = 2$ is 2.**

Clearly $1 < 2$ i.e. Geometric Multiplicity < Algebraic Multiplicity **(i)**

Putting $\lambda = 3$ in $(\lambda I - A) X = O$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is characteristic vector of A associated with characteristic root $\lambda = 3$

$\Rightarrow (3I - A) X = O$

$$\Rightarrow \begin{bmatrix} 0 & -10 & -5 \\ 2 & 6 & 4 \\ -3 & -5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to \frac{1}{2}R_2$

$$\Rightarrow \begin{bmatrix} 0 & -10 & -5 \\ 1 & 3 & 2 \\ -3 & -5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 + 3R_2$

$$\Rightarrow \begin{bmatrix} 0 & -10 & -5 \\ 1 & 3 & 2 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -10 & -5 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to -\frac{1}{5}R_2, R_3 \to \frac{1}{2}R_3$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x+3y+2z=0$ and $2y+z=0$

$\Rightarrow x=-3y-2z$ and $z=-2y$

$\Rightarrow x=-3y+4y$ and $z=-2y$

$\Rightarrow x=y$ and $z=-2y$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ -2y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} y$$

$\Rightarrow (1, 1, -2)$ is L.I. characteristic vector associated to characteristic root $\lambda = 3$

Since there is one L.I. characteristic vector associated to characteristic root $\lambda = 3$, so that the geometric multiplicity of $\lambda = 3$ is one. Also $\lambda = 3$ is a root of characteristic equation with multiplicity 1
so algebraic multiplicity of $\lambda = 3$ is 1.
$\Rightarrow$ Geometric multiplicity = Algebraic Multiplicity **(ii)**
Combining (i) & (ii), Geometric Multiplicity $\leq$ Algebraic Multiplicity
Hence result is verified.

14. Show that any two vectors corresponding to two distinct characteristic roots of a Hermitian matrix are orthogonal. (September 2009, April 2009)

Sol. Let X_1, X_2 be two characteristic vectors corresponding to two distinct characteristic roots λ_1, λ_2 of a Hermitian matrix A.

Then $AX_1 = \lambda_1 X_1$ **(1)**

and $AX_2 = \lambda_2 X_2$ **(2)**

$\because$ A is Hermitian $\therefore \lambda_1, \lambda_2$ are real.

Now
$$\lambda_1 X_2^\theta X_1 = X_2^\theta(\lambda_1 X_1)$$
$$= X_2^\theta(AX_1) \quad \text{[by (1)]}$$
$$= \left(X_2^\theta A^\theta\right) X_1 \quad [\because A^\theta = A]$$
$$= (AX_2)^\theta X_1$$
$$= (\lambda_2 X_2)^\theta X_1 \quad \text{[by (2)]}$$
$$= \bar{\lambda}_2 X_2^\theta X_1$$
$$= \lambda_2 X_2^\theta X_1 \quad [\because \lambda_2 \text{ is real}]$$

$\therefore \quad \lambda_1 X_2^\theta X_1 = \lambda_2 X_2^\theta X_1$

$\Rightarrow (\lambda_1 - \lambda_2) X_2^\theta X_1 = O$

But $\lambda_1 - \lambda_2 \neq 0$

$\therefore \quad X_2^\theta X_1 = O$

$\Rightarrow$ X_1 and X_2 are orthogonal.

15. Find all the eigen values and a basis of each eigen space of matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 9 \end{bmatrix}$$ **Is A diagonalizable? Give reasons. (April 2009)**

Sol. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 9 \end{bmatrix}$

The eigen values of A are the values of t such that

$|tI - A| = 0$

$$\Rightarrow \begin{bmatrix} (t-1) & -2 & -3 \\ 0 & (t-4) & -6 \\ 0 & -4 & (t-9) \end{bmatrix} = 0$$

$$\Rightarrow (t-1)\begin{bmatrix} 1 & -2 & -3 \\ 0 & (t-4) & -6 \\ 0 & -4 & (t-9) \end{bmatrix} = 0 \qquad \text{Expand by } C_1$$

$$\Rightarrow (t-1)\begin{vmatrix} (t-4) & -6 \\ -4 & (t-9) \end{vmatrix} = 0$$

$$\Rightarrow (t-1)\left[(t-4)(t-9) - 24\right] = 0$$

$$\Rightarrow (t-1)\left[t^2 - 13t + 12\right] = 0$$

$$\Rightarrow (t-1)(t-1)(t-12) = 0$$

or $(t-1)^2(t-12) = 0$

$\Rightarrow t = 1, 1, 12.$

Thus, $t = 1, 12$ are the two distinct eigen values of A.

To find basis of eigen space of $t = 1$:

Putting $t = 1$ in $(tI - A)X = O$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the eigen vector of A associated to eigen value $t = 1$

$$\therefore \begin{bmatrix} 0 & -2 & -3 \\ 0 & -3 & -6 \\ 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 0 & -2 & -3 \\ 0 & -3 & -6 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ we have $-2y - 3z = 0, -3y - 6z = 0, -2z = 0$

$\Rightarrow y = 0,\ z = 0$ and x can have any non-zero value

$\therefore \{(1,0,0)\}$ is basis of eigen space of $t = 1$

To find basis of eigen space of $t = 12$:

Putting $t = 12$ in $(tI - A)X = O$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector of A associated to eigen value t = 12.

$$\Rightarrow \begin{bmatrix} 11 & -2 & -3 \\ 0 & 8 & -6 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Operate } R_2 \to R_2 + 2R_3$$

$$\Rightarrow \begin{bmatrix} 11 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Operate } R_3 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 11 & -2 & -3 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 11x - 2y - 3z = 0 \Rightarrow 3z = 11x - 2y$

$-4y + 3z = 0 \Rightarrow 3z = 4y \quad \Rightarrow z = \frac{4}{3}y$

$\therefore$ we have $6y = 11x \quad \Rightarrow \quad x = \frac{6}{11}y$

$\therefore$ y is free variable

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ 1 \\ \frac{4}{3} \end{bmatrix} y.$$

$\Rightarrow \left\{\left(\frac{6}{11}, 1, \frac{4}{3}\right)\right\}$ is a basis of eigen space of $t = 12$

Since A has two L.I. eigen vectors namely.

(1, 0, 0) and $\left(\frac{6}{11}, 1, \frac{4}{3}\right)$ but not three.

Therefore, A is not diagonalizable.

Part - C

1

NOTION OF PROBABILITY

1. **A bag contains 5 balls. Two balls are drawn and found to be red. What is the probability of all the balls being red?** **(September 2013)**

Sol. After knowing the colour of the two balls, it is clear that the bag may contain 2, 3, 4 or 5 red balls.

$\Rightarrow n(S)=4$

If A is the event that the bag contains 5 red balls, the required probability is

$$P(A)=\frac{1}{4}$$

2. **For any n events $E_1, E_2, ..., E_n$ in a sample space, show that:**

$$P\left(\bigcup_{i=1}^{n} E_i\right)=\sum_{i=1}^{n} P(E_i)-\sum_{\substack{i\neq j\\1\le i,j\le n}} P(E_i\cap E_j)+\sum_{\substack{i\neq j\neq k\\1\le i,j,k\le n}} P(E_i\cap E_j\cap E_k)$$

$$+.........+(-1)^n P(E_1\cap E_2\cap ....\cap E_n).$$ **(April 2013)**

Sol. We shall prove the result by induction on n for n = 2, we have

$$P(E_1\cup E_2)=P(E_1)+P(E_2)-P(E_1\cap E_2)$$

which is always true

$\therefore$ the result holds for n = 2

Let the result holds for n = r

i.e. $$P\left(\bigcup_{i=1}^{r} E_i\right)=\sum_{i=1}^{r} P(E_i)-\sum_{\substack{i\neq j\\1\le i,j\le r}} P(E_i\cap E_j)+\sum_{\substack{i\neq j\neq k\\1\le i,j,k\le r}} P(E_i\cap E_j\cap E_k)$$

$$+..........+(-1)^{r-1} P(E_1\cap E_2..........\cap E_r)$$

is true

We shall now show that result is also true for n = r + 1

Consider

$$P\left(\bigcup_{i=1}^{r+1} E_i\right) = P\left(\left(\bigcup_{i=1}^{r} E_i\right) \cup E_{r+1}\right)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r+1} E_i\right) = P\left(\bigcup_{i=1}^{r} E_i\right) + P(E_{r+1}) - P\left(\left(\bigcup_{i=1}^{r} E_i\right) \cap E_{r+1}\right)$$

$$= P\left(\bigcup_{i=1}^{r} E_i\right) + P(E_{r+1}) - P\left(\bigcup_{i=1}^{r} (E_i \cap E_{r+1})\right)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r+1} E_i\right) = \sum_{i=1}^{r} P(E_i) - \sum_{\substack{i \neq j \\ 1 \le i,j \le r}} P(E_i \cap E_j) + \sum_{\substack{i \neq j \neq k \\ 1 \le i,j,k \le r}} P(E_i \cap E_j \cap E_k)$$

$$+ (-1)^{r-1} P(E_1 \cap E_2 ------ E_r) + P(E_{r+1}) - P\left(\bigcup_{i=1}^{r} (E_i \cap E_{r+1})\right)$$

$$= \sum_{i=1}^{r+1} P(E_i) - \sum_{\substack{i \neq j \\ 1 \le i,j \le r}} P(E_i \cap E_j) + \sum_{\substack{i \neq j \neq k \\ 1 \le i,j,k \le r}} P(E_i \cap E_j \cap E_k)$$

$$+ ------- + (-1)^{r-1} P(E_1 \cap E_2 ------ E_r)$$

$$- \left[\sum_{i=1}^{r} P(E_i \cap E_{r+1}) - \sum_{\substack{i \neq j \\ 1 \le i,j \le r}} P(E_i \cap E_{r+1} \cap E_j) + ------\right.$$

$$\left. + ---- (-1)^{r-1} P(E_1 \cap E_2 -- \cap E_{r+1})\right]$$

$$= \sum_{i=1}^{r+1} P(E_i) - \left[\sum_{\substack{i \neq j \\ 1 \le i,j \le r}} P(E_i \cap E_j) + \sum_{i=1}^{r} P(E_i \cap E_{r+1})\right]$$

$$+ \left[\sum_{\substack{i \neq j \neq k \\ 1 \le i,j,k \le r}} P(E_i \cap E_j \cap E_k) + \sum_{\substack{i \neq j \\ 1 \le i,j \le r}} P(E_i \cap E_{r+1} \cap E_j)\right]$$

$$+ ------ + (-1)^{r} P(E_1 \cap E_2 ---- \cap E_{r+1})$$

$$= \sum_{i=1}^{r+1} P(E_i) - \sum_{\substack{i \neq j \\ 1 \le i,j \le (r+1)}} P(E_i \cap E_j) + \sum_{\substack{i \neq j \neq k \\ 1 \le i,j,k \le (r+1)}} P(E_i \cap E_j \cap E_k)$$

$$+ ------ + (-1)^{(r+1)-1} P(E_1 \cap E_2 --- \cap E_{r+1})$$

$\therefore$ Result is true for n = r + 1

$\therefore$ By principle of mathematical induction the result is true $\forall\, n \in N$

3. State and prove Baye's Theorem. **(September 2012, April 2011, Sept. 2009)**

Sol. Statement:

If $E_1, E_2, \ldots .. E_n$ are mutually disjoint events with $P(E_i) \neq 0$, $i = 1,2,\ldots.,n$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that P(A) > 0, we have

$$P(E_i \mid A) = \frac{P(E_i) P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i) P(A \mid E_i)} \quad \text{where } i = 1, 2, \ldots, n$$

Proof:

We are given that

$$A \subset \bigcup_{i=1}^{n} E_i$$

$\therefore$ We have using distributive law

$$A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i)$$

Since $(A \cap E_i) \subset E_i, (i = 1,2,\ldots\ldots\ldots,n)$

and E_i 's is being mutually exclusive

$\therefore$ $A \cap E_i$'s are also mutually exclusive

$$P(A) = P\bigcup_{i=1}^{n} (A \cap E_i) = P\left[(A \cap E_1) \cup (A \cap E_2) \cup \ldots.. \cup (A \cap E_n)\right]$$

$$\Rightarrow P(A) = P(A \cap E_1) \cup (A \cap E_2) \cup \ldots.. \cup (A \cap E_n)$$

and using multiplication theorem of probability, we get

$$P(A) = P(E_1) P(A \mid E_1) + P(E_2) P(A \mid E_2) + \ldots. + P(E_n) P(A \mid E_n)$$

$$\Rightarrow P(A) = \sum_{i=1}^{n} P(E_i) P(A \mid E_i) \qquad (1)$$

Also, we have

$$P(A \cap E_i) = P(A) P(E_i \mid A) = P(E_i) P(A \mid E_i) \qquad \text{(By def)}$$

$$\Rightarrow P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} \qquad (2)$$

and using equation (1) in (2), we get

$$P(E_i \mid A) = \frac{P(A \cap E_i)}{\sum_{i=1}^{n} P(E_i) P(A \mid E_i)} \qquad (3)$$

$$\Rightarrow P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)}$$

Hence proved.

4. If $E_1, E_2, \ldots\ldots\ldots, E_n$ are n events, prove that $P\left(\bigcap_{i=1}^{n} E_1\right) > \sum_{i=1}^{n} P(E_i) - (n-1)$.

(April 2012)

Sol. **(i)** For any two events E_1, E_2

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow P(E_1 \cap E_2) = P(E_1) + P(E_2) - P(E_1 \cup E_2)$ **(1)**

By axiom (1) of probability

$P(E) \le 1 \Rightarrow \quad -P(E) \ge -1$

and in particular $-P(E_1 \cup E_2) \ge -1$

$\Rightarrow -\left[P(E_1) + P(E_2) - P(E_1 \cap E_2)\right] \ge -1$

$\Rightarrow P(E_1 \cap E_2) \ge P(E_1) + P(E_2) - 1$ **(2)**

$\Rightarrow$ The result is true for $n = 2$

Let us suppose that the result is true for $n = r$

i.e. $P\left(\bigcap_{i=0}^{r} E_i\right) \ge \sum_{i=1}^{r} P(E_i) - (r-1)$

Now consider n = r + 1

$P\left(\bigcap_{i=1}^{r+1} E_i\right) = P\left(\left(\bigcap_{i=1}^{r} E_i\right) \cap E_{r+1}\right) \ge P\left(\bigcap_{i=1}^{r} E_i\right) + P(E_{r+1}) - 1$ [Using (2)]

$\ge \sum_{i=1}^{r} P(E_i) - (r-1) + P(E_{r+1}) - 1$ [Using (3)]

$= \sum_{i=1}^{r} P(E_i) - (r+1-1)$

i.e. $P\left(\bigcap_{i=1}^{r} E_i\right) \ge \sum_{i=1}^{r+1} P(E_i) - r$

$\Rightarrow$ the result is true for $n = r+1$

Thus by mathematical induction the result is true $\forall n \in N$.

5. A and B will throw dice for a prize which is to be won by the player who first throws 6. If A throws the first, what are their respective expectations?

(September 2011)

Sol. Let the prize amount be Rs X

Game is to be won by the player who first throws 6

$\Rightarrow$ probability of success $= p = \frac{1}{6}$

Probability of failure $= q = \frac{5}{6}$

P(A wins) $= P\left(A + \bar{A}\bar{B}A + \bar{A}\bar{B}\bar{A}\bar{B}A + ---\right) = p + q^2p + q^4p + ------$

$= \frac{p}{1-q^2} = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{1}{6} \cdot \frac{36}{11} = \frac{6}{11}$

P(B wins) $= P\left(\bar{A}B + \bar{A}\bar{B}\bar{A}B + \bar{A}\bar{B}\bar{A}\bar{B}\bar{A}B + ---\right)$

$= qp + q^3p + q^5p + ------$

$= \frac{qp}{1-q^2} = \frac{\frac{5}{36}}{1 - \frac{25}{36}} = \frac{5}{36} \cdot \frac{36}{11} = \frac{5}{11}$

$\Rightarrow$ Expectation of $A = \frac{6}{11}X$

Expectation of $B = \frac{5}{11}X$

Ratios of expectation of A : B = 6 : 5

6. A bowl contains 10 chips, of which 8 are marked \$2 each and 2 are marked \$5 each. Let a person choose at random and without replacement 3 chips from the bowl. If the person is to receive the sum of the resulting amount, find his expectations. **(September 2011)**

Sol. A person chooses 3 chips from the bowl. Hence the possible amount in 3 draws would be 2 + 2 + 2 = 6 or 2 + 2 + 5 = 9 or 2 + 5 + 5 = 12

Amount of \$6 is possible in one way = (2, 2, 2)

Amount of \$9 is possible in three way = (2, 2, 5), (2, 5, 2), (5, 2, 2)

Amount of \$12 is possible in three ways = (2, 5, 5), (5, 2, 5), (5, 5, 2)

$\Rightarrow$ Expected Amount

$= 6\left(\left(\frac{8}{10}\right)\left(\frac{7}{9}\right)\left(\frac{6}{8}\right)\right) + 9\left(\left(\frac{8}{10}\right)\left(\frac{7}{9}\right)\left(\frac{2}{8}\right) + \left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{7}{8}\right) + \left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{7}{8}\right)\right)$

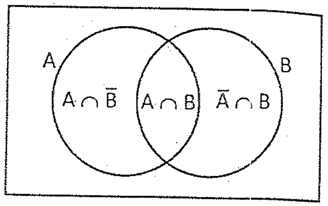
$$+12\left(\left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\left(\frac{1}{8}\right)+\left(\frac{2}{10}\right)\left(\frac{8}{9}\right)\left(\frac{1}{8}\right)+\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{8}{8}\right)\right)$$

$$=\frac{39}{5}=7.8$$

7. **Define Sample Space. If A and B are two events, then prove that $P(A \cup B) = P(A) + A(B) - P(A \cap B)$.** **(April 2011)**

Sol. **Sample Space:** The set of all possible out comes of a random experiment is called the sample space associated with that experiment for example: if we toss a die then sample space is $\{1,2,3,4,5,6\}$

Let A and B are two events



$\Rightarrow (\bar{A}\cap B)\cap(A\cap B)=\phi$

and $(\bar{A}\cap B)\cup(A\cap B)=B$

$\Rightarrow P[(\bar{A}\cap B)\cup(A\cap B)]=P(B)$

$\Rightarrow P(\bar{A}\cap B)+P(A\cap B)=P(B)$

$\Rightarrow P(\bar{A}\cap B)=P(B)-P(A\cap B)$

Similarly $P(A\cap \bar{B})=P(A)-P(A\cap B)$

Also $(\bar{A}\cap B)\cap(A\cap B)\cap(A\cap \bar{B})=\phi$

And $(\bar{A}\cap B)\cup(A\cap B)\cup(A\cap \bar{B})=(A\cup B)$

$\Rightarrow P[(\bar{A}\cap B)\cup(A\cap B)\cup(A\cap \bar{B})]=P(A\cup B)$

$\Rightarrow P(\bar{A}\cap B)+P(A\cap B)+P(A\cap \bar{B})=P(A\cup B)$

$\Rightarrow P(B)-P(A\cap B)+P(A\cap B)+P(A)-P(A\cap B)=P(A\cup B)$

$\Rightarrow P(A\cup B)=P(A)+P(B)-P(A\cap B)$

8. **A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that the ball drawn is (i) red (ii) white (iii) blue (iv) red or white.**

Sol. Box contains 6 red balls, 4 white balls and 5 blue balls.

(i) P(red ball) $= \frac{{}^{6}C_1}{{}^{15}C_1}=\frac{6}{15}=\frac{2}{5}$

(ii) P(white ball) $= \frac{{}^{4}C_1}{{}^{15}C_1}=\frac{4}{15}$

(iii) P(blue ball) $= \frac{{}^{5}C_1}{{}^{15}C_1}=\frac{5}{15}=\frac{1}{3}$

(iv) P(red or white ball) $= \frac{{}^{10}C_1}{{}^{15}C_1}=\frac{10}{15}=\frac{2}{3}$

9. **A car manufacturing factor has two plants. Plants P manufactures 70% of cars and Plant Q manufactures 30%. At Plant P 80% of cars are rated of standard quality and at Plant Q 90% of cars are rated of standard quality. A car is picked up at random and is found to be of standard quality. What is the probability that it has come from Plant P?** **(September 2010)**

Sol. Let E_1 denote the event that car manufactured by plant P and E_2 denote the event that car manufactured by plant Q.

Further A denotes the event that car manufactured is of standard quality

Accordingly

$P(E_1)=\frac{7}{10}$ $\qquad$ $P(E_2)=\frac{3}{10}$

$P(A|E_1)=\frac{8}{10}$ $\qquad$ $P(A|E_2)=\frac{9}{10}$

We need $P(E_1|A)$

According to Baye's Theorem,

$$P(E_1|A)=\frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1)+P(A|E_2)P(E_2)}$$

$$=\frac{\left(\frac{8}{10}\right)\left(\frac{7}{10}\right)}{\left(\frac{8}{10}\right)\left(\frac{7}{10}\right)+\left(\frac{9}{10}\right)\left(\frac{3}{10}\right)}$$

$$=\frac{56}{56+27}=\frac{56}{83}$$

10. **A coin is tossed (m + n) times (m > n). Find the probability of atleast m consecutive heads.** **(April 2010)**

Sol. Since m > n, only one sequence of m consecutive heads is possible

$\Rightarrow$ sequence may start with first toss or second toss and so on and last one will be starting with $(n+1)^{th}$ toss

Let E_i denote the event that sequence of m consecutive head starts with i^{th} toss.

$\Rightarrow$ Required probability is

$P(E_1)+P(E_2)+-----+P(E_{n+1})$

$P(E_1)=P$ [consecutive heads in first m tosses and head or tail in the rest]

$=\left(\frac{1}{2}\right)^m$

$P(E_r)=P$ [tail in the $(r-1)^{th}$ trial followed by m consecutive heads and head or tail in the rest]

$=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^m \qquad r=2,3------n+1$

Required Probability $=\frac{1}{2^m}+\frac{n}{2^{m+1}}=\frac{2+n}{2^{m+1}}$

11. A girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die? (April 2009)

Sol. Let A be the event of getting 5 or 6 and B the event of getting 1, 2, 3 or 4 when a die is thrown.

$P(A)=\frac{2}{6}=\frac{1}{3}$ and $P(B)=\frac{4}{6}=\frac{2}{3}$

Let H represents the event of getting a head when a coin is tossed.

When a coin is tossed thrice the conditional probability of getting a head

$P(H|A)={}^nC_r p^r.q^{n-r}={}^3C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2=\frac{3}{8}$

When a coin is tossed once the conditional probability of getting a head

$=P(H|B)=\frac{1}{2}$

$\therefore$ total probability of getting a head is

$P(H)=P(A).P(H|A)+P(B).P(H|B)$

$=\frac{1}{3}.\frac{3}{8}+\frac{2}{3}.\frac{1}{2}=\frac{1}{8}+\frac{1}{3}=\frac{11}{24}$

$\therefore$ the probability that a head comes when 1, 2, 3 or 4 come on the die.

$=\frac{P(B).P(H|B)}{P(H)}=\frac{\frac{2}{3}.\frac{1}{2}}{\frac{11}{24}}=\frac{1}{3}\times\frac{24}{11}=\frac{8}{11}$ *Ans.*

2

SKEWNESS AND KURTOSIS

1. Find the first four moments:
(i) about the origin
(ii) about mean.
For a random variable X having p.d.f.

$f(x)=\frac{3}{4}x(2-x),\ 0\le x\le 2$

Is the distribution symmetrical about the mean? (September 2013)

Sol. (i) μ_r' (about the origin) $=\int_a^b x^r f(x)\,dx$

$\therefore$ μ_1' (about the origin) $=\frac{3}{4}\int_0^2 x^2(2-x)\,dx$

$=\frac{3}{4}\int_0^2(2x^2-x^3)\,dx=\frac{3}{4}\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2$

$=\frac{3}{4}\left[\frac{16}{3}-\frac{16}{4}\right]=\frac{3}{4}\times16\times\frac{1}{12}=1$

μ_2' (about the origin) $=\frac{3}{4}\int_0^2 x^3(2-x)\,dx$

$=\frac{3}{4}\int_0^2(2x^3-x^4)\,dx=\frac{3}{4}\left[\frac{2x^4}{4}-\frac{x^5}{5}\right]_0^2$

$=\frac{3}{4}\left[\frac{32}{4}-\frac{32}{5}\right]=\frac{3}{4}\times32\times\frac{1}{20}=\frac{6}{5}$

μ_3' (about the origin) $==\frac{3}{4}\int_0^2 x^4(2-x)\,dx$

$$= \frac{3}{4}\int_0^2 (2x^4 - x^5)\,dx = \frac{3}{4}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{64}{5} - \frac{64}{6}\right] = \frac{3}{4}\times 64 \times \frac{1}{30} = \frac{8}{5}$$

μ_4' (about the origin) $= = \frac{3}{4}\int_0^2 x^5(2-x)\,dx$

$$= \frac{3}{4}\int_0^2 (2x^5 - x^6)\,dx = \frac{3}{4}\left[\frac{2x^6}{6} - \frac{x^7}{7}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{128}{6} - \frac{128}{7}\right] = \frac{3}{4}\times 128 \times \frac{1}{42} = \frac{16}{7}$$

(ii) μ_r (about the mean)

Using Recursive Relation for central moments and moments about origin

$\mu_1 = 0$ (first central moment is always zero)

$$\mu_2 = \mu_2' - \mu_1'^2 = \frac{6}{5} - (1)^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \frac{8}{5} - 3\left(\frac{6}{5}\right)(1) + 2(1)^3 = \frac{8}{5} - \frac{18}{5} + 2 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = \frac{16}{7} - 4\left(\frac{8}{5}\right)(1) + 6\left(\frac{6}{5}\right)(1)^2 - 3(1)^4$$

$$= \frac{16}{7} - \frac{32}{5} + \frac{36}{5} - 3 = \frac{3}{35}$$

Since μ_3 which is measure of skewness is zero, the distribution is symmetrical about the mean

2. **Find the coefficient of:** **(September 2013)**
(i) Skewness
(ii) Kurtosis
For the distribution having probability density function.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, -\infty < x < \infty.$$

Sol. Given $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad -\infty < x < \infty$

μ_1' (about origin) $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{\frac{-x^2}{2}}\,dx$

$= 0$ $\left[\because xe^{\frac{-x^2}{2}} \text{ is odd function}\right]$

Similarly, $\mu_3' = 0$

$$\mu_2' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{2}}\,dx$$

$= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{2}}\,dx$ $\left[\because x^2 e^{\frac{-x^2}{2}} \text{ is even function}\right]$

Put $\frac{x^2}{2} = t \Rightarrow x\,dx = dt \Rightarrow dx = \frac{dt}{\sqrt{2t}}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (2t) e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t}\,dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}-1} e^{-t}\,dt = \frac{2}{\sqrt{\pi}} \Gamma\frac{3}{2} = \frac{2}{\sqrt{\pi}}\left(\frac{3}{2}\right)\Gamma\frac{1}{2}$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{3}{2}\right)\left(\sqrt{\pi}\right) = 3$$

Similarly $\mu_4' = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (2t)^2 e^{-t} \frac{dt}{\sqrt{2t}}$

$$= \frac{4}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}} e^{-t}\,dt$$

$$= \frac{4}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{5}{2}-1} e^{-t}\,dt = \frac{4}{\sqrt{\pi}} \Gamma\frac{5}{2} = \frac{4}{\sqrt{\pi}}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\frac{1}{2}$$

$$= \frac{4}{\sqrt{\pi}}\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\sqrt{\pi}\right) = 15$$

Now

$\mu_1 = 0$ (first central moment is always zero)

$$\mu_2 = \mu_2' - \mu_1'^2 = 3 - (0)^2 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0 - 3(3)(0) + 2(0)^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 15 - 4(0)(0) + 6(3)(0)^2 - 3(0)^4 = 15$$

(i) Coefficient of skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = 0 \qquad \left[\because \ \sigma^2 = \mu_2\right]$$

(ii) Coefficient of kurtosis

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{15}{(3)^2} - 3 = -\frac{4}{3}$$

3. **Find coefficient of (i) Skewness (ii) Kurtosis for the Poisson distribution.** **(April 2013)**

Sol. Coefficient of skewness $= \frac{\mu_3}{\sigma^3}$

Coefficient of Kurtosis $= \frac{\mu_4}{\sigma^4} - 3$

For Poisson distribution

Mean = Variance = λ

$$\mu_2 = \mu_2' - \mu_1'^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = (\lambda^3 + 3\lambda^2 + \lambda) - 3(\lambda)(\lambda^2 + \lambda) - 2\lambda^3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) + 4\lambda(\lambda^3 + 3\lambda^2 + \lambda)$$

$$+ 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4 = 3\lambda^2 + \lambda$$

Coefficient of skewness $= \frac{\mu_3}{\sigma^3} = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\sqrt{\lambda}}$

Coefficient of Kurtosis $= \frac{\mu_4}{\sigma^4} - 3 = \frac{3\lambda^2 + \lambda}{\lambda^2} - 3 = \frac{1}{\lambda}$

4. **Let first, second and third moments of a random variable about point 7 be 3, 11 and 15 respectively. Find mean and then first, second and third moments about mean.** **(September 2011)**

Sol. Given $\mu_1' = E(X-7) = 3$ (1)

$\mu_2' = E(X-7)^2 = 11$ (2)

$\mu_3' = E(X-7)^3 = 15$ (3)

From (1) $E(X) - E(7) = 3$

$\Rightarrow E(X) - 7 = 3$

$\Rightarrow E(X) = 10 \quad \Rightarrow \quad$ Mean = 10

From (2) $E(X^2) - 14E(X) + E(49) = 11$

$\Rightarrow E(X^2) - 140 + 49 = 11$

$\Rightarrow E(X^2) = 102$

From (3) $E(X^3) - 21E(X^2) + 147E(X) - E(343) = 15$

$\Rightarrow E(X^3) - 21(102) + 147(10) - 343 = 15$

$\Rightarrow E(X^3) = 1030$

Moments about mean

$\mu_1 = E(X-10) = E(X) - E(10) = 10 - 10 = 0$

$\mu_2 = E(X-10)^2 = E(X^2) - 20E(X) + E(100)$

$= 102 - 20(10) + 100$

$= 2$

$\mu_3 = E(X-10)^3 = E(X^3) - 30E(X^2) + 300E(X) - E(1000)$

$= 1030 - 30(102) + 300(10) - 1000$

$= -30$

5. **Find the measures of skewness and kurtosis of the Poisson distribution with mean μ.** **(September 2011)**

Sol. We know,

Coefficient of skewness $= \frac{\mu_3}{\sigma^3}$

Coefficient of Kurtosis $= \frac{\mu_4}{\sigma^4} - 3$

M.G.F. of poisson distribution with mean μ is given by $M_X(t) = e^{\mu(e^t - 1)}$

$$\mu_1' = \frac{d}{dt}\left|M_X(t)\right|_{t=0} = \mu$$

$$\mu_2' = \frac{d^2}{dt^2}\left|M_X(t)\right|_{t=0} = \mu^2 + \mu$$

$$\mu_3' = \frac{d^3}{dt^3}\left|M_X(t)\right|_{t=0} = \mu^3 + 3\mu^2 + \mu$$

$$\mu_4' = \frac{d^4}{dt^4}\left|M_X(t)\right|_{t=0} = \mu^4 + 6\mu^3 + 7\mu^3 + \mu$$

Now central moments

$$\mu_2 = \mu_2' - \mu_1'^2 = \mu$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \mu$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 3\mu^2 + \mu$$

Now coefficient of skewness $= \frac{\mu_3}{\sigma^3} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{\mu}{\mu^{\frac{3}{2}}} = \frac{1}{\sqrt{\mu}}$

Coefficient of skewness $= \frac{\mu_4}{\sigma^4} - 3 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{3\mu^2 + \mu}{\mu^2} - 3 = \frac{1}{\mu}$

3

RANDOM VARIABLES

1. **If the density function of a random variable X is given by:**

$$f(x) = \begin{cases} x & ; \ 0 < x < 1 \\ 2 - x & ; \ 1 \le x < a \\ 0 & ; \ \text{elsewhere} \end{cases}$$

Find:

(i) The value of a

(ii) The distributive function of x

(iii) $P(0.8 < x < 0.6a)$. **(September 2013)**

Sol. Since $f(x)$ is a density function

(i) $\int_0^a f(x)\,dx = 1$

$\Rightarrow \int_0^1 x\,dx + \int_1^a (2-x)\,dx = 1$

$\Rightarrow \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^a = 1$

$\Rightarrow \left[\frac{1}{2} - 0\right] + \left[\left(2a - \frac{a^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right] = 1$

$\Rightarrow \frac{1}{2} + 2a - \frac{a^2}{2} - \frac{3}{2} = 1$

$\Rightarrow a^2 - 4a + 4 = 0$

$\Rightarrow (a-2)^2 = 0$

$\Rightarrow a = 2, 2$

Hence value of $a = 2$

(ii) The cumulative distribution function of X is

$$F(x)=\begin{cases}\int_{-\infty}^{x} 0\,dt=0 & x\le 0\\ \int_{-\infty}^{0} 0\,dt+\int_{0}^{x} t\,dt=\frac{x^2}{2} & 0<x<1\\ \int_{-\infty}^{0} 0\,dt+\int_{0}^{1} t\,dt+\int_{1}^{x}(2-t)\,dt=-\frac{x^2}{2}+2x-1 & 1\le x<2\\ \int_{-\infty}^{0} 0\,dt+\int_{0}^{1} t\,dt+\int_{1}^{2}(2-t)\,dt+\int_{2}^{x} 0\,dt=1 & x\ge 2\end{cases}$$

(iii) $P(0.8<X<0.6a)=P(0.8<X<1.2)$

$$=\int_{0.8}^{1.2} f(x)\,dx=\int_{0.8}^{1} x\,dx+\int_{1}^{1.2}(2-x)\,dx$$

$$=\left[\frac{x^2}{2}\right]_{0.8}^{1}+\left[2x-\frac{x^2}{2}\right]_{1}^{(1.2)}$$

$$=\left[\frac{1}{2}-\frac{(0.8)^2}{2}\right]+\left[2(1.2)-\frac{(1.2)^2}{2}-2+\frac{1}{2}\right]$$

$$=[0.5-0.32]+[2.4-.72-2+0.5]$$

$$=0.36$$

2. Define Moment generating function of a random variable

(September 2013)

Sol. (i) Moment Generating function of a random variate X (about origin) is given by

$$M_X(t)=E\left(e^{tx}\right)=\begin{cases}\int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{for continuous random variable X}\\ \sum_{x} e^{tx} P(X=x), & \text{for discrete random vairable X}\end{cases}$$

3. The probability density function of a random variable is given by

$$f(x)=\begin{cases}4x\left(1-x^2\right) & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$ **find:**

(i) Mean (ii) Mode (iii) Median of x.

(April 2013)

Sol. (i) Mean $=\int_{-\infty}^{\infty} x f(x)\,dx$

$\Rightarrow$ Mean $=\int_{0}^{1} 4x^2\left(1-x^2\right)dx$

$$=\int_{0}^{1}\left(4x^2-4x^4\right)dx=\left[\frac{4x^3}{3}-\frac{4x^5}{5}\right]_{0}^{1}$$

$$=\left[\frac{4}{3}-\frac{4}{5}\right]=\frac{8}{15}$$

(ii) Let M be the median

$$\Rightarrow \int_{-\infty}^{M} f(x)\,dx=\int_{M}^{\infty} f(x)\,dx=\frac{1}{2}$$

$$\Rightarrow \int_{0}^{M} 4x\left(1-x^2\right)dx=\frac{1}{2}$$

$$\Rightarrow \left[2x^2-x^4\right]_{0}^{M}=\frac{1}{2}$$

$$\Rightarrow 2M^2-M^4=\frac{1}{2}$$

$$\Rightarrow 2M^4-4M^2+1=0$$

$$\Rightarrow 2\left(M^4-2M^2+1\right)-1=0$$

$$\Rightarrow 2\left(M^2-1\right)^2=1$$

$$\Rightarrow M^2-1=\pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow M^2=1\pm\frac{1}{\sqrt{2}}$$

But $0\le x\le 1$ $\quad\Rightarrow 0\le M\le 1$

$$\Rightarrow M^2=1-\frac{1}{\sqrt{2}}$$

Or $M=\sqrt{1-\frac{1}{\sqrt{2}}}=0.5412$

(iii) Mode

Mode is the value where f(x) is maximum

$$\Rightarrow f'(x)=0$$

$$\Rightarrow 4-12x^2=0$$

$$\Rightarrow x^2=\frac{1}{3}$$

$\Rightarrow x = \frac{1}{\sqrt{3}}$

Hence Mode $= \frac{1}{\sqrt{3}}$

4. **If $f(x) = cx^2, 0 < x < 1$ is p.d.f. of a continuous random variable X. Find**

(September 2012)

(i) $P\left(\frac{1}{3} < X < 2\right)$ **(ii) Find 'a' such that** $P(X \le a) = P(X > a)$

Sol. (i) $f(x)$ is a probability density function if $f(x) \ge 0$ which is true for all values of x

and $c \ge 0$ and $\int_0^1 f(x)dx = 1 \Rightarrow c\int_0^1 x^2 dx = 1 \Rightarrow c = 3$

Thus the prob. density function becomes

$f(x) = 3x^2$

(ii) $P\left(\frac{1}{3} < X < \frac{1}{2}\right) = \int_{1/3}^{1/2} f(x)dx = \int_{1/3}^{1/2} 3x^2 dx = \left[x^3\right]_{1/3}^{1/2} = \frac{1}{8} - \frac{1}{27} = \frac{19}{216}$

(iii) Given $P(X \le a) = P(X > a)$ and $a \in (0,1)$

$\Rightarrow \int_0^a 3x^2 dx = \int_a^1 3x^2 dx$

$\Rightarrow a^3 = 1 - a^3 \Rightarrow 2a^3 = 1$

$\Rightarrow a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$

5. **Let X be the number of gallons of ice-cream that is required at a certain store on a hot summer day. Let** **(April 2012)**

$$f(x) = \begin{cases} \frac{12x(100-x)^2}{10^{12}} & , 0 < x < 1000 \\ 0 & , \text{elsewhere} \end{cases}$$

be the probability density function of X. How many gallons of ice-cream should the store have on hand each of these days, so that the probability of exhausting its supply on a particular day is 0.05?

Sol. Let 'c' gallons of ice-cream be required

Such that $P(X > c) = 0.05$

$P(X \le c) = .95$

$\int_0^c \frac{12}{10^{12}}\left(x(1000-x)^2\right)dx = .95$

$\frac{12}{10^{12}}\int_0^c \left(10^6 x - 2000x^2 + x^3\right)dx = .95$

$\frac{12}{10^{12}}\int_0^c \left[10^6\frac{x^2}{2} - 2000\frac{x^3}{3} + \frac{x^4}{4}\right]_0^c = .95$

$\frac{6}{10^6}c^2 - \frac{8}{10^9}c^3 + \frac{3}{10^{12}}c^4 = .95$

Solving Numerically

$c \approx 751.395$

6. **Define cumulative distribution of function and its important properties.**

(September 2010)

Sol. **(a) Definition:** The cumulative distribution function, or briefly the distribution function, for a random variable X is defined for all real x by

$F(x) = P(X \le x) = P\{\omega \in S : X(\omega) \le x\}$

where S is the Sample space and $-\infty < x < \infty$.

(b) Properties of Distribution Function:

Property I: The distribution function F(x) is non decreasing

$\left[i.e., F(x_1) \le F(x_2) \text{ if } x_1 < x_2\right]$

Property II: If F is the distribution function of a random variable X and if $a < b$, then

$P(a < X \le b) = F(b) - F(a)$.

Property III: If F is the distribution function of the random variable X and if $a < b$, then

$P(a \le X \le b) = P(X = a) + \left[F(b) - F(a)\right]$

Property IV: If F is the distribution function of the random variable X and if $a < b$, then,

$P(a < X < b) = F(b) - F(a) - P(X = b)$

Property V: If F is the distribution function of the random variable X and if $a < b$, then,

$P(a \le X < b) = F(b) - F(a) - P(X = b) + P(X = a)$

Property VI: For the distribution function F of a random variable X.

(i) $\lim_{x \to -\infty} F(x) = 0$

(ii) $\lim_{x \to \infty} F(x) = 1$

Property VII. $F(x)$ is continuous from the right

$\left[i.e., \lim_{h \to 0+} F(x+h) = F(x) \text{ for all } (x) \right]$.

7. **Let X be a continuous random variable with p.d.f. given by** **(September 2010)**

$$f(x) = \begin{cases} kx & ; \ 0 \le x < 1 \\ k & ; \ 1 \le x < 2 \\ -kx + 3k & ; \ 2 \le x < 3 \\ 0 & ; \ elsewhere \end{cases}$$

(i) Determine k

(ii) Determine f(x), the cumulative distribution function.

Sol. (i) $f(x)$ will be a prob. density function if

$f(x) \ge 0$, which is true for all givens values of x and $k \ge 0$

And $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{0} 0dx + \int_{0}^{1} kxdx + \int_{1}^{2} kdx + \int_{2}^{3} (-kx+3k)dx + \int_{3}^{\infty} 0dx = 1$$

$$\Rightarrow \left.\frac{kx^2}{2}\right|_0^1 + kx\Big|_1^2 + \left[-\frac{kx^2}{2} + 3kx \right]_2^3 = 1$$

$$\Rightarrow \frac{k}{2} + k(2-1) + \left(9k - \frac{9k}{2}\right) - (6k - 2k) = 1$$

$$\Rightarrow \frac{3k}{2} + \frac{9k}{2} - 4k = 1$$

$$\Rightarrow 2k = 1 \qquad \Rightarrow k = \frac{1}{2}$$

(ii) For $-\infty < x < 0$;

$$F(x) = \int_{-\infty}^{x} 0.dt = 0$$

For $0 \le x < 1$;

$$F(x) = \int_{-\infty}^{0} 0.dt + \int_{0}^{x} \frac{t}{2} dt = \frac{x^2}{4}$$

For $1 \le x < 2$;

$$F(x) = \int_{-\infty}^{0} 0.dt + \int_{0}^{1} \frac{t}{2} dt + \int_{1}^{x} \frac{1}{2} dt$$

$$= \left.\frac{t^2}{4}\right|_0^1 + \left.\frac{t}{2}\right|_1^x = \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4} = \frac{2x-1}{4}$$

For $2 \le x < 3$,

$$F(x) = \int_{-\infty}^{0} 0.dt + \int_{0}^{1} \frac{t}{2} dt + \int_{1}^{2} \frac{1}{2} dt + \int_{2}^{x} \left(-\frac{t}{2} + \frac{3}{2} \right) dt$$

$$= \frac{1}{4} + \left(1 - \frac{1}{2}\right) + \left(\frac{-x^2}{4} + \frac{3x}{2} - 2 \right) = \frac{-x^2}{4} + \frac{3x}{2} - \frac{5}{4}$$

For $3 \le x < \infty$,

$$F(x) = \int_{-\infty}^{0} 0.dt + \int_{0}^{1} \frac{t}{2} dt + \int_{1}^{2} \frac{1}{2} dt + \int_{2}^{3} \left(-\frac{t}{2} + \frac{3}{2} \right) dt + \int_{3}^{x} 0.dt$$

$$= \frac{1}{4} + \left(1 - \frac{1}{2}\right) + \left(-\frac{9}{4} + \frac{9}{2} + 1 - 3 \right) = 1$$

Hence, the distribution function F(x) is given by

$$F(x) = \begin{cases} 0, & \text{for } -\infty < x < 0 \\ \frac{x^2}{4}, & \text{for } 0 \le x < 1 \\ \frac{2x-1}{4}, & \text{for } 1 \le x < 2 \\ \frac{-x^2}{4} + \frac{3x}{2} - \frac{5}{4}, & \text{for } 2 \le x < 3 \\ 1, & \text{for } 3 \le x < \infty. \end{cases}$$

(iii) The probability that X is larger than 1.5 is given by:

$P(X > 1.5) = 1 - P(X \le 1.5)$

$= 1 - F(1.5)$

$= \left(1 - \frac{3-1}{4}\right) = \frac{1}{2}$

Also the probability that X is not larger than 1.5

$= P(X < 1.5) = \frac{2(1.5)-1}{4} = \frac{1}{2}$

Hence, out of the three numbers x_1, x_2 and x_3, the probability that exactly one is larger than 1.5 is:

$P(x_1).P(\bar{x}_2).P(\bar{x}_3)+P(\bar{x}_1).P(x_2).P(\bar{x}_3)+P(\bar{x}_1).P(\bar{x}_2).P(x_3)$

$=3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{3}{8}$

8. The probability mass function of a discrete random variable x is zero except at the point $x=0,1,2$. Also $p(0)=3c^3, p(1)=4c-10c^2$ and $p(2)=5c-1$, for some $c>0$.

(i) Determine the value of c.

(ii) Find the largest value of x such that $F(x)<\frac{1}{2}$.

(iii) Find the smallest value of x such that $F(x)>\frac{1}{3}$. Here $F(x)$ denotes the distribution function of x. **(April 2010)**

Sol. (i) To find c

$p(0)+p(1)+p(2)=1$

$3c^3+4c-10c^2+5c-1=1$

$3c^3-10c^2+9c-2=0$

$(c-1)(c-2)(3c-1)=0$

$\Rightarrow\ c=\frac{1}{3}$ because c = 1 and 2 will make $p(0)>1$ which is not possible.

(ii) P.m.f

X	0	1	2
P(X)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

C.d.f

X	0	1	2
F(X)	$\frac{1}{9}$	$\frac{1}{3}$	1

From distribution function it is clear largest value of x such that $F(x)<\frac{1}{2}$ is 1

(iii) From distribution function it is clear smallest value of x such that $F(x)>\frac{1}{3}$ is 2

9. **Define the distribution function of a continuous random variable.**

Verify that $f(x)=\begin{cases}0 & ;x<-a\\ \frac{1}{2}\left(\frac{x}{a}+1\right); & -a\le x\le a\\ 1 & ;x>a\end{cases}$ **is a distribution function.**

(September 2009)

Sol. **Definition:** The cumulative distribution function, or briefly the distribution function, for a random variable X is defined for all real x by

$F(x)=P(X\le x)=P\{\omega\in S: X(\omega)\le x\}$

where S is the sample space and $-\infty<x<\infty$

(i) $F_X(x)$ is non-decreasing and right continuous

and $F_X(-\infty)=0,\ F_X(\infty)=1$.

$\therefore\ F_X(x)$ satisfies properties of cumulative distribution. Hence $F_X(x)$ represents the distribution function of a random variable X.

(ii) The prob. Distribution function of random variable X is

$f(x)=\frac{d}{dx}F_X(x)=\frac{1}{2a}\ for -a\le X\le a$

$f(x)$ satisfies the conditions

(i) $f(x)\ge 0$, for given values of x

and (ii) $\int_{-a}^{a} f(x)\,dx=\int_{-a}^{a}\frac{1}{2a}\,dx=1$

$\therefore\ f(x)=\frac{1}{2a}; -a\le x\le a$ is Prob. distribution function and $F_X(x)$ is distribution function of random variable X.

10. **A random variable X has the probability density function**

$f(x)=\frac{a}{x^2+1}, -\infty<x<\infty$

(i) Find the value of the constant 'a'

(ii) Find $P\left(\frac{1}{3}\le X^2\le 1\right)$

(iii) Find the distribution function of X. **(April 2009)**

Sol. **(a)** By definition, for $f(x)$ to be a density function, we must have

(i) $f(x)\ge 0$

which is true for all values of x and $a \geq 0$

and (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e. $\int_{-\infty}^{\infty} \frac{a\,dx}{x^2+1} = a\left[\tan^{-1} x\right]_{-\infty}^{\infty} = a\left[\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right] = 1$

$\Rightarrow a.\pi = 1$

$\Rightarrow a = \frac{1}{\pi}$

(b) The Prob. distribution function is

$$f(x) = \frac{1}{\pi(x^2+1)}, -\infty < x < \infty$$

Now, if $\frac{1}{3} \leq X^2 \leq 1$,

Then $X^2 \leq 1 \Rightarrow |X| \leq 1$

$\Rightarrow -1 \leq X \leq 1$ **(1)**

And $X^2 \geq \frac{1}{3}$

$\Rightarrow |X| \geq \frac{1}{3}$

$\Rightarrow X \geq \frac{1}{\sqrt{3}}$ or $X \leq -\frac{1}{\sqrt{3}}$ **(2)**

Combining (1) and (2)

$\frac{\sqrt{3}}{3} \leq X \leq 1$ or $-1 \leq X \leq -\frac{\sqrt{3}}{3}$

Thus, the required probability is

$$P\left(\frac{1}{3} \leq X^2 \leq 1\right) = P\left(\frac{\sqrt{3}}{3} \leq X \leq 1 \quad \text{or} \quad -1 \leq X \leq -\frac{\sqrt{3}}{3}\right)$$

$$= P\left(\frac{\sqrt{3}}{3} \leq X \leq 1\right) + P\left(-1 \leq X \leq \frac{-\sqrt{3}}{3}\right)$$

$$= \frac{1}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1} + \frac{1}{\pi}\int_{-1}^{-\sqrt{3}/3} \frac{dx}{x^2+1}$$

$$= \frac{2}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1}$$

$$= \frac{2}{\pi}\left[\tan^{-1} x\right]_{\sqrt{3}/3}^{1}$$

$$= \frac{2}{\pi}\left[\tan^{-1}(1) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{6}$$

(c) The distribution function corresponding to the density function is

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \frac{1}{\pi}\int_{-\infty}^{x} \frac{du}{u^2+1}$$

$$= \frac{1}{\pi}\left[\tan^{-1} u\Big|_{-\infty}^{x}\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$$

$$\Rightarrow F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x.$$

4

DISCRETE RANDOM VARIABLES

1. **The moment generating function of a random variable X is given by $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$. Evaluate $P(\mu-2\sigma<X<\mu+2\sigma)$ where μ is the mean and σ the standard deviation of x.** **(April 2013)**

Sol. $M_X(t)=\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$

$\Rightarrow p=\frac{2}{3},\ q=\frac{1}{3},\ n=9$

$\mu=np=9\left(\frac{2}{3}\right)=6$

$\sigma^2=npq=9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)=2$

$P(\mu-2\sigma<X<\mu+2\sigma)=P\left(-2<\frac{X-\mu}{\sigma}<2\right)$

$\Rightarrow \frac{X-\mu}{\sigma}=Z$ a standard Normal Variate

$\Rightarrow P(\mu-2\sigma<X<\mu+2\sigma)=P(-2<Z<2)$

$=2P(0<Z<2)$

$=2(.4772)=.9544$

2. **Define Bernoulli's random variable** **(September 2013)**

Sol. Bernoulli's Random Variable: A random variable X which takes two values 0 and 1, with probabilities q and p respectively i.e. $P(X=1)=p$, $P(X=0)=q=1-p$ is called a Bernoulli variate and is said to have a Bernoulli distribution.

3. **Let the random variable X assume the value 'r' with the probability law: $P(X=r)=q^r.p,\quad r=0,1,2,3,\ldots\ldots$ Find the m.g.f. of X and hence its mean and variance.** **(September 2012, April 2009)**

Sol. $M_X(t)=E\left(e^{tX}\right)=\sum_{r=0}^{\infty} e^{tr}q^r.p$

$=P\sum_{r=0}^{\infty}\left(qe^t\right)^r$

$=P\left(1+qe^t+\left(qe^t\right)^2+\ldots\ldots\right)$

$=\frac{p}{1-qe^t}$

differentiating w.r.t. t, we have

$\frac{dM_X(t)}{dt}=\frac{pqe^t}{\left(1-qe^t\right)^2}$, and

$\frac{d^2M_X(t)}{dt^2}=pqe^t\frac{\left(1+qe^t\right)}{\left(1-qe^t\right)^3}$

$\therefore\ \mu_1'$ (about origin) $=\left.\frac{dM_X(t)}{dt}\right|_{t=0}$

$=\frac{pq}{(1-q)^2}=\frac{q}{p}$

$\therefore\ \mu_2'$ (about origin) $=\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}$

$=\frac{pq(1+q)}{(1-q)^3}=\frac{q(1+q)}{p^2}$

Hence, mean $=\mu_1'$ (about origin) $=\frac{q}{p}$

and variance $=\mu_2=\mu_2'-\mu_1'^2=\frac{q+q^2}{p^2}-\frac{q^2}{p^2}=\frac{q}{p^2}$.

4. **The m.g.f. of binomial variate about origin was found to be $\left(\frac{3}{4}+\frac{1}{4}e^t\right)^8$, find:**

(i) Mean, S.D. and Coefficient of Variation
(ii) Mode
(iii) Find P(X = 2). **(September 2012)**

Sol. We know if X ~ B (*n, p*), then its moment generating function is

$M_X(t)=(q+pe^t)^n$

Comparing with $\left(\frac{3}{4}+\frac{1}{4}e^t\right)^8$ $\quad \therefore q=\frac{3}{4}, p=\frac{1}{4}, n=8$

(i) Thus, Mean = $np=8\times\frac{1}{4}=2$,

Variance $=npq=8\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{2}$

SD = $\sqrt{\frac{3}{2}}$,

and Coefficient of variation = $\frac{SD}{Mean}=\frac{\sqrt{\frac{3}{2}}}{2}=.612$

(ii) The mode is given by $(n+1)p=(8+1)\times\frac{3}{4}=9\times\frac{3}{4}=\frac{27}{4}=6.75$ which is not an integer.

$\therefore$ The only Mode = Integer part of $(n+1)p=6$

$\therefore$ Mode = 6

(iii) $P(X=2)={}^8C_2p^2q^6$

$={}^8C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6=0.3115$

5. **If X has a Poisson distribution with parameter λ, show that the distribution function of X is given by**

$F(x)=\frac{1}{\Gamma(x+1)}\int_\lambda^\infty e^{-t}t^x dt$, x = 0, 1, 2,......... **(September 2012)**

Sol. If X is a Poisson variate, then

$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}; x=0,1,2,....$ (1)

Consider the incomplete gamma integral;

$I_x=\frac{1}{x!}\int_\lambda^\infty e^{-t}t^x dt$; (x is a positive integer)

$$=\left|\frac{-e^{-t}t^x}{x!}\right|_\lambda^\infty+\frac{1}{(x-1)!}\int_\lambda^\infty e^{-t}t^{x-1}dt=\frac{e^{-\lambda}\lambda^x}{x!}+I_{x-1} \quad (2)$$

Which is a reduction formula for I_x.

Repeated applications of (2) gives

$I_x=\frac{e^{-\lambda}\lambda^x}{x!}+\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}+....+\frac{e^{-\lambda}\lambda}{1!}+I_0$

But $I_0=\int_\lambda^\infty e^{-t}dt=\left|-e^{-t}\right|_\lambda^\infty=e^{-\lambda}$

$\therefore I_x=e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2e^{-\lambda}}{2!}+....+\frac{\lambda^x}{x!}e^{-\lambda}$

= P (X = 0) + (X = 1) + ... P (X = x). [From (1)]

= P (X ≤x) = F(x)

where F is the distribution function of Variable X.

$\Rightarrow F(x)=\frac{1}{x!}\int_\lambda^\infty e^{-t}t^x dt=\frac{1}{\Gamma(x+1)}\int_\lambda^\infty e^{-t}t^x dt$

6. **A communication system consists of n components, each of which functions independently with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what value of p is a 5-component system more likely to operate effectively than a 3-component system?** **(April 2012)**

Sol. Let X denotes number of components working effectively in n-component system.

So 5-component system will operate effective if $X\geq 3$

and in 3-component system $X\geq 2$

We need 5-component system to operate effectively more likely than 3-component system

$P(X\geq 3)>P(X\geq 2)$

(5-component) (3-component)

${}^5C_3p^3q^2+{}^5C_4p^4q+{}^5C_5p^5>{}^3C_2p^2q+{}^3C_3p^3$

$\Rightarrow 10p(1-p)^2+5p^2(1-p)+p^3>3(1-p)+p$

(Dividing both sides by p^2 and replacing q by $1-p$)

$\Rightarrow 10p+10p^3-20p^2+5p^2-5p^3+p^3>3-2p$

$\Rightarrow 6p^3-15p^2+12p-3>0$

$\Rightarrow 2p^3-5p^2+4p-1>0$

$\Rightarrow (2p-1)(p-1)^2>0$

$\Rightarrow p>\frac{1}{2}$

Hence a 5-component system will operate more effectively than a 3-component system if $p>\frac{1}{2}$

7. **Find the mean and variance of a geometric random variable.** **(April 2012)**

Sol. $M_X(t)=E(e^{tx})=\sum_{x=0}^{\infty}e^{tx}.P(X=x)=\sum_{x=0}^{\infty}e^{tx}.q^x p$

$=p\sum_{x=0}^{\infty}(q.e^t)^x=p\left\{1+(qe^t)+(qe^t)^2+(qe^t)^3+.......\right\}$

$=p\times\frac{1}{1-qe^t}$

$\therefore M_X(t)=\frac{p}{1-qe^t}$.

$\mu_1'=\left[\frac{dM_X(t)}{dt}\right]_{t=0}=\left[\frac{d}{dt}p(1-qe^t)^{-1}\right]_{t=0}=pq(1-q)^{-2}=\frac{q}{p}$

$\mu_2'=\left[\frac{d^2M_X(t)}{dt^2}\right]_{t=0}=\frac{q}{p}+\frac{2p^2}{p^2}$

$\therefore$ Variance $=\mu_2'-(\mu_1')^2=\frac{2q^2}{p^2}+\frac{q}{p}-\frac{q^2}{p^2}$

$=\frac{q^2}{p^2}+\frac{q}{p}=\frac{q^2+pq}{p^2}=\frac{q(p+q)}{p^2}=\frac{q}{p^2}$

8. **The moment generating function of a random variable X is exp $\left[\mu(e^t-1)\right]$. Find $P(\mu-2\sigma<X<\mu+2\sigma)$ where μ is the mean and σ the standard deviation of X.** **(April 2012)**

Sol. Given $M_X(t)=\exp\left[\mu(e^t-1)\right]$

$=e^{\mu(e^t-1)}$

$\Rightarrow$ X is poison random variable with parameter μ

Hence Mean $=\mu$

Variance $=\sigma^2=\mu$ $\Rightarrow \sigma=\sqrt{\mu}$

$P(\mu-2\sigma<x<\mu+2\sigma)=P\left(-2<\frac{x-\mu}{\sigma}<2\right)$

X is random variable with mean μ and standard deviation σ

$\Rightarrow z=\frac{x-\mu}{\sigma}$ is a standard normal variate

$P(\mu-2\sigma<x<\mu+2\sigma)=P(-2<z<2)$

$=2P(0<z<2)$

$=2(.4772)$

$=.9544$

9. **Define Bernoulli random variables.** **(April 2011)**

Sol. **Bernoulli Random variable:**

A random variable X which takes two values 0 and 1 with probabilities q and p respectively then X is Bernoulli Random variate

x	0	1
$p(x)$	q	p

Also $M_X(t)=q+pe^t$

Mean $=p$

Variance $=pq$

10. **Let X is a binomial variable based on n repetitions, then prove that:**

$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}, k=0,1,.....,n.$ **(April 2011)**

Sol. Consider set of n independent trials in which probability p of success in any trial is constant and q the probability of failure i.e. $q=1-p$.

The probability of k successes and $n-k$ failures in n independent trials in a specified order says $SSFSFF------FSF$ is given by compound probability theorem by expression

$$P(SSFSFF------FSF)=P(S)P(S)P(F)P(S)P(F)P(F)$$

$$---P(F)P(S)P(F)$$

$$=ppqpqq---qpq$$

$$=(\underbrace{pp-----p}_{k\text{ factors}})(\underbrace{qq----q}_{(n-k)\text{ factors}})=p^k q^{n-k}$$

But k successes in n trials can occur in ^nC_k ways.

Hence $P(X=k)={}^nC_k p^k q^{n-k}$

11. Show that in a Poisson distribution with unit mean, mean deviation about mean is $\left(\frac{2}{e}\right)$ times the standard deviation. **(September 2010)**

Sol. Here, we are given $\lambda = 1$

$$\therefore\ P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-1}.1}{x!}=\frac{e^{-1}}{x!};x=0,1,2,....$$

Mean deviation about mean 1 is

$$E(|X-1|)=\sum_{x=0}^{\infty}|x-1|p(x)$$

$$=e^{-1}\sum_{x=0}^{\infty}\frac{|x-1|}{x!}$$

$$=e^{-1}\left[1+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+......\right]$$

We have

$$\frac{n}{(n+1)!}=\frac{(n+1)-1}{(n+1)!}=\frac{1}{n!}-\frac{1}{(n+1)!}$$

$\therefore$ Mean deviation about mean

$$=e^{-1}\left[1+\left(1-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{3!}-\frac{1}{4!}\right)+......\right]$$

$$=e^{-1}(1+1)=\frac{2}{e}\times 1$$

$$=\frac{2}{e}\times \text{standard deviation,}$$

Since for the poisson distribution

Variance = Mean = λ = 1 (given).

12. If a random variable X follows binomial distribution with parameters n and p, prove that P(X is even) = $\frac{1}{2}\left[1+(q-p)^n\right]$. **(April 2010)**

Sol. We know

$$(q+p)^n=\left[{}^nC_0q^n+{}^nC_1q^{n-1}p+{}^nC_2q^{n-2}p^2+--+{}^nC_{n-1}qp^{n-1}+{}^nC_np^n\right]$$

$$(q-p)^n=\left[{}^nC_0q^n+{}^nC_1q^{n-1}(-p)+{}^nC_2q^{n-2}(-p)^2+--+{}^nC_{n-1}q(-p)^{n-1}+{}^nC_n(-p)^n\right]$$

Adding the equations

$$\left[(q+p)^n+(q-p)^n\right]=2\left[{}^nC_0q^n+{}^nC_2q^{n-2}p^2+{}^nC_4q^{n-4}p^4+--\right]$$

$$=2\left[{}^nC_0q^n+{}^nC_2p^2q^{n-2}+{}^nC_4p^4q^{n-4}+--\right]$$

Let X be binomial variate with parameter n and p

We know $P(X=r)={}^nC_rp^rq^{n-r}$

From above

$$\left[P(X=0)+P(X=2)+P(X=4)+---\right]=\frac{1}{2}\left[(q+p)^n+(q-p)^n\right]$$

$$P(X=even)=\frac{1}{2}\left[1+(q-p)^n\right]\qquad \because q+p=1$$

13. Let X and Y be two independent Poisson Variates. Show that X + Y is also a Poisson Variate. Does the result hold if X and Y are correlated? Justify your answer. **(April 2010)**

Sol. Let X and Y be two independent Poisson Variates $X\sim P(\lambda_1)$ and $Y\sim P(\lambda_2)$

$$M_{X+Y}(t)=M_X(t)M_Y(t)$$

$$=e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)}$$

$$=e^{(\lambda_1+\lambda_2)(e^t-1)}$$

which is the moment generating function of a Poisson variate with parameter $\lambda_1+\lambda_2$

$$\Rightarrow X+Y\sim P(\lambda_1+\lambda_2)$$

Also if X and Y are correlated

$$\therefore\ E(X+Y)=E(X)+E(Y)=\lambda_1+\lambda_2$$

$$\text{var}(X+Y)=\text{var}\,X+\text{var}\,Y+2\,\text{cov}(X,Y)$$

$= \lambda_1 + \lambda_2 + 2\text{cov}(X,Y)$

$\Rightarrow E(X+Y) \neq \text{var}(X+Y)$

Hence X and Y are correlated then X + Y is not poisson variate because for poisson variate mean and variance are equal.

14. Obtain the moment generating function of the Binomial distribution with $n = 7$, $p = 0.6$. Find the first three moments of the distribution. **(September 2009)**

Sol. M.G.F of Binomial distribution

$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\left({}^nC_x p^x q^{n-x}\right)$

$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} = (q + pe^t)^n$

$\mu_1' = \frac{d}{dt}|M_X(t)|_{t=0} = np$

$\mu_2' = \frac{d^2}{dt^2}|M_X(t)|_{t=0} = np(np+q)$

$\mu_3' = \frac{d^3}{dt^3}|M_X(t)|_{t=0} = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

Given $n = 7$ $p = 0.6 \Rightarrow q = 0.4$

$\mu_1' = np = 7(.6) = 4.2$

$\mu_2' = np(np+q) = 7(.6)(7(.6)+.4) = 19.32$

$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$= (7)(6)(5)(.6)^3 + 3(7)(6)(.6)^2 + 7(.6)$

$= 45.36 + 45.36 + 4.2$

$= 94.92$

15. If X and Y are independent Poisson Variates, show that the conditional distribution of X given $X+Y$ is binomial. **(September 2009)**

Sol. Let X and Y be two independent poisson variables with parameters m_1 and m_2 respectively. So $X+Y$ is poisson with parameter $m_1 + m_2$

$$P(X = r | X+Y = n) = \frac{P(X = r \cap X+Y = n)}{P(X+Y = n)} = \frac{P(X = r).P(Y = n-r)}{P(X+Y = n)}$$

$$= \frac{e^{-m_1} m_1^r}{\lfloor r} \cdot \frac{e^{-m_2} m_2^{n-r}}{\lfloor n-r} \cdot \frac{\lfloor n}{e^{-(m_1+m_2)}(m_1+m_2)^n}$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r}\left(\frac{m_1}{m_1+m_2}\right)^r \left(\frac{m_2}{m_1+m_2}\right)^{n-r}$$

$= {}^nC_r P^r Q^{n-r}$ where $P = \frac{m_1}{m_1+m_2}$ $Q = \frac{m_2}{m_1+m_2}$

$\Rightarrow$ Distribution of X given X + Y is binomial

16. What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability of success in each trial? **(September 2009)**

Sol. Define X as the random variable which gives number of failures preceding first success

$\therefore$ X = 0, 1, 2, 3, ...

Let $p =$ probability of success, $q =$ probability of failure such that

$p + q = 1$

$\therefore$ The prob. mass function of X is

X	0	1	2	3	4	....
$p(x)$	p	qp	q^2p	q^3p	q^4p	....

$E(X) = \sum_{x=0}^{\infty} x.P(X = x) = \sum_{x=0}^{\infty} x.(q)^x p$

$= pq\sum_{x=1}^{\infty} xq^{x-1}$

$= pq\left[1 + 2q + 3q^2 + 4q^3 + ...\right]$

$= pq \times \frac{1}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$

$E(X) = \frac{q}{p}$.

17. If X is Poisson variate with parameter m and μ_r is the rth central moment, prove that $m\left[{}^rC_1\mu_{r-1} + {}^rC_2\mu_{r-2} + ... {}^rC_r\mu_0\right] = \mu_{r+1}$ **(April 2009)**

Sol. Let $X \sim P(m)$; so $P(X = x) = \frac{e^{-m} m^x}{\lfloor x}$

$\mu_{r+1} = E\left[(X - E(X))\right]^{r+1} = E\left[(X - m)\right]^{r+1}$

$$=\sum_{x=0}^{\infty}(x-m)^{r+1}P(X=x)$$

$$=\sum_{x=0}^{\infty}(x-m)(x-m)^r\frac{e^{-m}m^x}{\lfloor x}$$

$$=\sum_{x=0}^{\infty}\frac{x(x-m)^r e^{-m}m^x}{\lfloor x}-m\sum_{x=0}^{\infty}(x-m)^r\frac{e^{-m}m^x}{\lfloor x}$$

$$=\sum_{x=0}^{\infty}\frac{(x-m)^r e^{-m}m^x}{\lfloor x-1}-m\mu_r \qquad \text{Put } y=x-1$$

$$=\sum_{y=0}^{\infty}\frac{(y-m+1)^r e^{-m}m^{y+1}}{\lfloor y}-m\mu_r$$

$$=m\sum_{y=0}^{\infty}(y-m+1)^r.P(Y=y)-m\mu_r$$

$$=m\sum_{y=0}^{\infty}\Big[(y-m)^r+{}^rC_1(y-m)^{r-1}+{}^rC_2(y-m)^{r-2}+....$$

$$......+{}^rC_{r-1}(y-m)+{}^rC_r(y-m)^0\Big]P(Y=y)-m\mu_r$$

$$=m\left[\mu_r+{}^rC_1\mu_{r-1}+{}^rC_2\mu_{r-2}.......+{}^rC_r\mu_0\right]-m\mu_r$$

$$\therefore\ \mu_{r+1}=m\left({}^rC_1\mu_{r-1}+{}^rC_2\mu_{r-2}.......+{}^rC_r\mu_0\right)$$

5

CONTINUOUS RANDOM VARIABLES

1. Define

Rectangular random variable **(September 2013)**

Sol. Rectangular Random Variable: A random variable X is said to be a continuous rectangular random variable over interval (a,b) if its p.d.f. is given by:

$$f(x)=\begin{cases}\dfrac{1}{b-a}, & a<x<b\\ 0\ , & \text{otherwise}\end{cases}$$

2. If X and Y are independent gamma variates, show that $X+Y$ and $\dfrac{X}{X+Y}$ are independently distributed. **(September 2013)**

Sol. Since X is a $\gamma(\mu)$ variate and Y is a $\gamma(v)$ variate, we have

$$f_1(x)dx=\frac{1}{\Gamma(\mu)}e^{-x}x^{\mu-1}dx;\ 0<x<\infty,\ \mu>0$$

$$f_2(y)dy=\frac{1}{\Gamma(v)}e^{-y}y^{v-1}dy;\ 0<y<\infty,\ v>0.$$

Since X and Y are independently distributed, their joint probability differential is given by the compound probability theorem as known below:

$$dF(x,y)=f_1(x)f_2(y)dxdy$$

$$=\frac{1}{\Gamma(\mu)\Gamma(v)}e^{-(x+y)}x^{\mu-1}y^{v-1}dxdy$$

Now $u=x+y, z=\dfrac{x}{x+y}$, so that $x=uz, y=u-x=u(1-z)$

The Jacobian of the transformation J is given by:

$$J=\frac{\partial(x,y)}{\partial(u,z)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z}\end{vmatrix}=\begin{vmatrix} z & 1-z\\ u & -u\end{vmatrix}=-u.$$

As X and Y range from 0 to ∞ and z from 0 to 1 $\left(\because \frac{x}{x+y}\le 1\right)$.

Hence the joint distribution of U and Z is given by:

$$dG(u,z)=g(u,z)\,dudz=\frac{1}{\Gamma(\mu)\Gamma(v)}e^{-u}(uz)^{\mu-1}\left[u(1-z)\right]^{v-1}|J|\,dudz$$

$$=\frac{1}{\Gamma(\mu)\Gamma(v)}.e^{-u}.u^{\mu+v-1}z^{\mu-1}(1-z)^{v-1}\,dudz$$

$$=\left\{\frac{e^{-u}u^{-\mu+v-1}}{\Gamma(\mu+v)}du\right\}\left\{\frac{1}{\beta(\mu,v)}z^{\mu-1}(1-z)^{v-1}dz\right\}$$

$$=\left[g_1(u)du\right]\left[g_2(z)dz\right], \qquad \text{(say)} \qquad (1)$$

$$\left.\begin{aligned} &\textit{where} \quad g_1(y)=\frac{1}{\Gamma(\mu+v)}e^{-u}.u^{\mu+v-1},0<u<\infty\\ &\textit{and} \quad g_2(z)=\frac{1}{\beta(\mu,v)}z^{\mu-1}.(1-z)^{v-1},0<z<1\end{aligned}\right\} \qquad (2)$$

From (i) and (ii), we conclude that **U and Z are independently distributed, U as $\gamma(\mu+v)$ variate and Z as a $\beta_1(\mu,v)$ variate.**

3. **Assume the mean height of soldiers to be 68.22 inches and a variance of 10.8 inches square. How many soldiers in a regiment of 10,000 would you expect to be over 6 feel tall, given that the area under the standard normal curve between x = 0 and x = 0.35 is 0.1368 and between x = 0 and x = 1.5 is 0.3746? (April 2013)**

Sol. Let X denote height of soldiers

Given $\mu=68.22 \quad \sigma^2=10.8$

$$P(X>72)=P\left(\frac{X-68.22}{\sqrt{10.8}}>\frac{72-68.22}{\sqrt{10.8}}\right)$$

$$=P(Z>1.15)=0.5-P(0<Z<1.15)$$

$$=0.5-0.3749$$

$$=0.1251$$

⇒ Expected number of soldiers having height above 6 feet

$$=1000\times 0.1251$$

$$=125.1\approx 125$$

4. **Define a rectangular distribution. Find its mean and variance. If X is a rectangular random variable with mean 1 and standard deviation $\frac{2}{\sqrt{3}}$, find P(X< 0). (April 2013)**

Sol. A random variable X is said to be continuous rectangular distribution over interval [a, b] If its p.d.f is given by

$$f(x)=\begin{cases}\frac{1}{b-a} & a\le x\le b\\ 0 & \text{otherwise}\end{cases}$$

$$\text{Mean}=\mu_1'=\int_a^b xf(x)\,dx=\int_a^b\frac{x}{b-a}dx$$

$$=\frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)=\frac{b+a}{2}$$

$$\text{Mean}=\frac{b+a}{2}$$

$$\mu_2'=\int_a^b x^2 f(x)\,dx=\int_a^b\frac{x^2}{b-a}dx$$

$$=\frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)=\frac{1}{3}\left(b^2+ab+a^2\right)$$

$$\text{Variance}=\mu_2'-\mu_1'^2=\frac{1}{3}\left(b^2+ab+a^2\right)-\left(\frac{1}{2}(b+a)\right)^2$$

$$=\frac{1}{12}(b-a)^2$$

Given Mean = 1

$$\frac{a+b}{2}=1 \qquad \Rightarrow a+b=2$$

$$\text{S.D.}=\frac{2}{\sqrt{3}}$$

$$\frac{1}{12}(b-a)^2=\frac{4}{3}\Rightarrow b-a=4$$

$$\Rightarrow b=3,\ a=-1$$

$$P(X<0)=\int_{-1}^{0}\frac{1}{4}dx=\left[\frac{x}{4}\right]_{-1}^{0}=\frac{1}{4}$$

5. Prove that $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ **is p.d.f.** **(September 2012)**

Sol. To show $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is p.d.f

We need to show $\int_{-\infty}^{\infty} f(x)dx=1$

Consider $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$

Put $\frac{x-\mu}{\sigma}=z$

$x=\mu+\sigma z$

$dx=\sigma dz$

$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}}\sigma dz$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}}dz$

$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{\frac{-z^2}{2}}dz$

Put $\frac{z^2}{2}=\lambda \Rightarrow z^2=2\lambda$

$z=\sqrt{2}\sqrt{\lambda}$

$dz=\sqrt{2}\frac{1}{2\sqrt{\lambda}}d\lambda$

$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\lambda}\frac{1}{\sqrt{2}\sqrt{\lambda}}d\lambda$

$=\frac{\sqrt{2}}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\lambda}\lambda^{\frac{-1}{2}}d\lambda=\frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-\lambda}\lambda^{\frac{1}{2}-1}d\lambda$

$=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)$

$=\frac{1}{\sqrt{\pi}}\sqrt{\pi}=1$

Hence given $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is a p.d.f.

6. Define a Normal random variable. Write the main properties of the normal distribution and normal probability curve. **(April 2012)**

Sol. **(a) Definition:** A random variable X is said to be a normal random variable with parameter μ (called 'mean') and σ^2 (called 'variance') if its probability density function is given by the law:

$$f(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

or $f(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$-\infty<x<\infty, -\infty<\mu<\infty, \sigma>0$.

When a random variable is normally distributed with mean μ and standard deviation σ, it is customary to write that X is distributed as $N(\mu,\sigma^2)$ and is expressed by $X\sim N(\mu,\sigma^2)$.

Chief Characteristics of the Normal Distribution and Normal Probability Curve: The normal probability curve with mean μ and standard deviation σ is given by the equation:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty<x<\infty$$

and has the following properties:

(i) The curve is bell-shaped and symmetrical about the line $x=\mu$.

(ii) Mean, Median and Mode of the distribution coincide and are each equal to μ.

(iii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x=\mu$, and is given by:

$$[p(x)]_{max}=\frac{1}{\sigma\sqrt{2\pi}}$$

(iv) $\beta_1=0$ and $\beta_2=3$

(v) $\mu_{2r+1}=0, (r=0,1,2,.....)$ and $\mu_{2r}=1.3.5.....(2r-1)\sigma^{2r}, (r=0,1,2,......)$

(vi) Since $f(x)$ being the probability, can never be negative, no portion of the curve lies below the x-axis.

(vii) Linear combination of independent normal variates is also a normal variate.

(viii) x-axis is an asymptote to the curve.

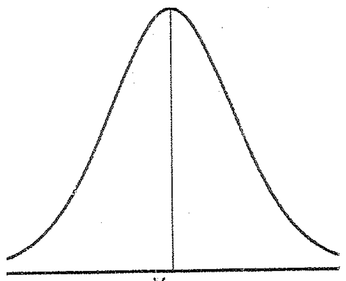
(ix) The points of inflexion of the curve are:

$x = \mu \pm \sigma, f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}}$.

(x) Area property.

$P(\mu - \sigma < X < \mu + \sigma) = 0.6826, P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$.

and $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$



7. **Write a short note on Gamma random variable.** **(April 2012)**

Sol. (a) **Definition:** A random variable X is said to be a gamma random variable with parameter $\lambda > 0$, if its prob. distribution function is given by:

$$f(x) = \begin{cases} \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}; & \lambda > 0, 0 < x < \infty \\ 0 \quad ; & \text{otherwise} \end{cases} \qquad (1)$$

X is known as a Gamma variate with parameter λ and referred to as a $\gamma(\lambda)$ variate.

Here $\Gamma(\gamma) = \int_0^\infty e^{-x} x^{\lambda-1} dx; \lambda > 0$

is called the gamma integral.

The function $f(x)$ defined above represents a probability function, since

$$\int_0^\infty f(x)\,dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-x} x^{\lambda-1} dx = \frac{1}{\Gamma(\lambda)} . \Gamma(\lambda) = 1$$

A continuous random variable X having the following prob. distribution function is said to have a gamma distribution with two parameter a and λ.

$$f(x) = \begin{cases} \frac{e^{\lambda}}{\Gamma(\lambda)} e^{-ax} . x^{\lambda-1}; a > 0, \lambda > 0, 0 < x < \infty \\ 0 \quad ; \quad \text{otherwise} \end{cases} \qquad (2)$$

Here $X \sim \gamma(a, \lambda)$. Taking a = 1 in (2) we get (1). Hence we may write $X \sim \gamma(\lambda) = \gamma(1, \lambda)$.

(i) The cumulative distribution function, called incomplete gamma function is defined as:

$$F_X(x) = \begin{cases} \int_0^x f(u)\,du = \frac{1}{\Gamma(\lambda)} \int_0^x e^{-u} . u^{\lambda-1} du, x > 0 \\ 0 \quad ; \quad \text{otherwise} \end{cases}$$

(ii) Moment generating function of Gamma random variable

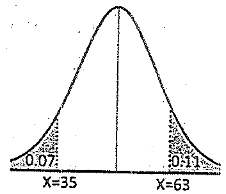
$M_X(t) = (1-t)^{-\lambda}, \quad |t| < 1$

(iii) Mean and variance of Gamma distribution both are λ

$\Rightarrow$ Mean = Variance = λ

8. **In normal distribution, 7% items are below 35 and 89% are below 63. Find mean and standard deviation of the distribution.** **(September 2011)**

Sol. Let μ be the mean and σ be the standard deviation



$z_1 = \frac{64 - \bar{x}}{\sigma}$

$\Rightarrow p(0 < z < z_1) = 0.5 - 0.11 = .39$

z_1 corresponding to .39 is 1.23

$\Rightarrow \frac{64 - \mu}{\sigma} = 1.23 \qquad (1)$

$-z_2 = \frac{35 - \bar{x}}{\sigma}$

$\Rightarrow P(0 < z < z_2) = 0.5 - 0.07 = .43$

z_2 corresponding to .43 is 1.48

$\Rightarrow \frac{35 - \mu}{\sigma} = -1.48 \qquad (2)$

From (1) and (2)

$64 - \mu = 1.23\sigma$

$35 - \mu = -1.48\sigma$

$\Rightarrow \sigma = 10.70$

$\Rightarrow \mu = 50.84$

9. If families are selected randomly in a certain thickly populated area and their monthly income in excess of Rs. 4000 is treated as exponential random variable with parameter $\frac{1}{2000}$ find the probability that 3 out of 4 families selected have income in excess of Rs. 5000. **(September 2011)**

Sol. $f(x)=\lambda e^{-\lambda x}=\frac{1}{2000}e^{\frac{-x}{2000}}; x>0$

x is the number of persons whose annual income is in excess of 4000

$x=5000-4000=1000$

$$P(x>1000)=\int_{1000}^{\infty}\frac{1}{2000}e^{\frac{-x}{2000}}dx=e^{\frac{-1}{2}}=\frac{1}{\sqrt{e}}$$

$$p=P(x>1000)=\frac{1}{\sqrt{e}}$$

$$q=P(x\le 1000)=1-\frac{1}{\sqrt{e}}$$

Probability that 3 out of 4 have income in excess of Rs 5000

$$={}^4C_3p^3q=4\left(\frac{1}{\sqrt{e}}\right)^3\left(1-\frac{1}{\sqrt{e}}\right)=\frac{4\left(\sqrt{e}-1\right)}{e^2}$$

10. A doctor recommends a patient to go on a particular diet for two weeks and there is equal likelihood for the patient to lose his weight between 2 kg and 4 kg. What is the average amount the patient is expected to loose on this diet? **(April 2011)**

Sol. Let X be the random variable defining the weight that the patient will loose on this particular diet. This random variable follows uniform distribution with p.d.f. as follows:

$$f(x)=\begin{cases}\frac{1}{2} & ; \quad 2\le x\le 4\\ 0 & ; \quad otherwise\end{cases}$$

We know for uniform distribution expectation is given by

$$E(X)=\frac{b+a}{2}$$

Here a = 2, b = 4

∴ average amount the patient is expected to loose on this diet is

$$\left(\frac{4+2}{2}\right)=3kgs.$$

11. Define Exponential random variables. **(April 2011)**

Sol. Exponential Random Variable:

A random variable X is said to follow exponential distribution with parameter λ (λ> 0) and its p.d.f. is given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & ; \quad x>0\\ 0 & ; \quad \text{otherwise}\end{cases}$$

Also $M_X(t)=\frac{\lambda}{\lambda-t}$

Mean = $\frac{1}{\lambda}$

Variance = $\frac{1}{\lambda^2}$

12. Write a short note on Gamma random variable **(April 2011)**

Sol. Gamma Random Variable: A random variable X is said to be a gamma random variable with parameter λ> 0, if its p.d.f. is given by

$$f(x)=\begin{cases}\frac{e^{-x}x^{\lambda-1}}{\overline{|\lambda}} & ; \quad \lambda>0,\ 0<x<\infty\\ 0 & ; \quad \text{otherwise}\end{cases}$$

Also $M_X(t)=\frac{1}{(1-t)^{\lambda}}, |t|\le 1$

Mean = λ

Variance = λ

13. Find the m.g.f. of Normal distribution and hence find its expectation and variance. **(September 2010)**

Sol. M.G.F. of normal distribution (about origin) is given by

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

Put $\frac{x-\mu}{\sigma}=z$

$\Rightarrow x=\mu+\sigma z$

$\Rightarrow dx=\sigma dz$

$$\Rightarrow M_X(t)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{t(\mu+\sigma z)}e^{-\frac{z^2}{2}}\sigma dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \sigma \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((z-\sigma t)^2 - \sigma^2 t^2)} dz$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z-\sigma t)^2} dz$$

Put $z - \sigma t = u$

$dz = du$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-\frac{u^2}{2}} du$$

Put $\frac{u^2}{2} = v \qquad \Rightarrow u^2 = 2v$

$$\Rightarrow du = \frac{dv}{\sqrt{2}\sqrt{v}}$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} 2\int_0^{\infty} e^{-v} \frac{dv}{\sqrt{2}\sqrt{v}}$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \sqrt{2} \int_0^{\infty} e^{-v} v^{-1/2} dv = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \sqrt{2} \left|\frac{1}{2}\right.$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \sqrt{2}\sqrt{\pi} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Hence $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Now

$$\mu_1' = Mean = \frac{d}{dt}|M_X(t)|_{t=0} = \left| e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t) \right|_{t=0} = \mu$$

$$\mu_2' = \frac{d^2}{dt^2}|M_X(t)|_{t=0} = \left| e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t)^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2} (\sigma^2) \right|_{t=0}$$

$$= \mu^2 + \sigma^2$$

Now variance = $\mu_2' - \mu_1'^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

14. A random variable X has exponential distribution with parameter λ = 3. Find P (X ≥ 4). S.D. and coefficient of variation. (September 2010)

Sol. Let X ~ exp (3)

$\therefore f(x) = 3e^{-3x}; x > 0$

(i) $P(X > 4) = \int_4^{\infty} f(x)dx = \int_4^{\infty} 3e^{-3x} dx = \left[-e^{-3x}\right]_4^{\infty}$

$= -\left[0 - e^{-12}\right] = \frac{1}{e^{12}}$

(ii) $E(X) = \int_0^{\infty} xf(x)dx = 3\int_0^{\infty} xe^{-3x} dx$

$= 3\left[\frac{xe^{-3x}}{-3} - \left(\frac{e^{-3x}}{9}\right)\right]_0^{\infty} = 3\left[0 - \left\{0 - \frac{1}{9}\right\}\right] = \frac{1}{3}$

also $E(X^2) = \int_0^{\infty} x^2 f(x)dx = 3\int_0^{\infty} x^2 e^{-3x} dx$

$= 3\left[x^2\left(\frac{e^{-3x}}{-3}\right) - 2x\left(\frac{e^{-3x}}{9}\right) + 2\left(\frac{e^{-3x}}{-27}\right)\right]_0^{\infty}$

$= 3\left[0 - \left\{0 - 0 + \frac{2}{-27}\right\}\right] = \frac{2}{9}$

$\therefore$ Variance = $E(X^2) - (E(X))^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

and thus S.D. = $\sqrt{Variance} = \frac{1}{3}$

$\therefore$ C.V. = $\frac{SD}{Mean} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$

15. If X has a uniform distribution on [0, 1]. Find the distribution function of $-2\log_e X$. (April 2010)

Sol. $X \sim U(0,1) \qquad \Rightarrow f(x) = 1 \;;\; 0 \le x \le 1$

Let $Y = -2\log_e X$

Now c.d.f of random variable Y is given by

$G_Y(y) = P(Y \le y) = P(-2\log_e X \le y) = P\left(\log_e X \ge \frac{-y}{2}\right)$

$$= P\left(X \ge e^{\frac{-y}{2}}\right) = 1 - P\left(X \le e^{\frac{-y}{2}}\right)$$

$$= 1 - \int_0^{e^{-y/2}} f(x)\,dx = 1 - \int_0^{e^{-y/2}} 1.dx = 1 - e^{-y/2}$$

$$G_Y(y) = 1 - e^{-y/2}$$

p.d.f. of random variable Y is given by

$$g_Y(y) = \frac{d}{dy}(G_Y(y)) = \frac{1}{2}e^{-y/2} \quad ; \quad 0 < y < \infty$$

which is exponential distribution with parameter $\frac{1}{2}$

16. Show that for a normal distribution **(April 2010)**

$\mu_{2n} = (2n-1)\mu_{2n-2}\sigma^2$ and $\mu_{2n-1} = 0$ where the symbols have their usual meaning.

Sol. M.G.F. (about mean) of Normal distribution is given by

$$E\left(e^{t(X-\mu)}\right) = e^{-\mu t}E\left(e^{tX}\right) = e^{-\mu t}M_X(t)$$

$$= e^{-\mu t}.e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$= e^{\frac{1}{2}\sigma^2 t^2}$$

M.G.F. (about mean) = $= e^{\frac{1}{2}\sigma^2 t^2}$

$$= \left[1 + \left(\frac{\sigma^2 t^2}{2}\right) + \frac{1}{2!}\left(\frac{\sigma^2 t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{\sigma^2 t^2}{2}\right)^3 + - - - -\right]$$

The coefficient of $\frac{t^r}{r!}$ gives u_r, the r^{th} moment about mean.

Also there is no term with odd powers of t

$\Rightarrow$ all moments of odd order about mean vanishes

i.e. $\mu_{2n+1} = 0 \quad ; \quad n = 0, 1, 2 - - - - - - -$

$\Rightarrow \mu_{2n}$ = coefficient of $\frac{t^{2n}}{(2n)!}$

$$= \frac{\sigma^{2n} \times (2n)!}{2^n n!}$$

$$= \frac{(2n)(2n-1)\sigma^2}{2n}\left[\frac{\sigma^{2n-2}(2n-2)!}{2^{n-1}(n-1)!}\right]$$

$$= (2n-1)\sigma^2[\mu_{2n-2}]$$

$$= (2n-1)\mu_{2n-2}\sigma^2$$

17. Define a Gamma Variate. If X and Y are independent Gamma Variates with parameter u and v respectively, show that X + Y and $\frac{X}{X+Y}$ are also independent. **(April 2010)**

Sol. Gamma distribution: The continuous random variable X which is distributed according to probability law:

$$f(x) = \begin{cases} \frac{e^{-x}x^{\lambda-1}}{\overline{|\lambda}} & ; \quad \lambda > 0,\ 0 < x < \infty \\ 0 & \quad \text{otherwise} \end{cases}$$

X is Gamma variate with parameter λ

Since X and Y are independent Gamma variates with parameters u and v respectively

$$f_1(x)dx = \frac{1}{\overline{|(u)}}e^{-x}x^{u-1}dx \quad ; \quad 0 < x < \infty,\ u > 0$$

$$f_2(y)dy = \frac{1}{\overline{|(v)}}e^{-y}y^{v-1}dy \quad ; \quad 0 < y < \infty,\ v > 0$$

Their joint probability differential is given by the compound probability theorem as

$$dF(x,y) = f_1(x)f_2(y)\,dxdy = \frac{1}{\overline{|u}\overline{|v}}e^{-(x+y)}x^{u-1}y^{v-1}dxdy$$

Let $a = x + y \qquad b = \frac{x}{x+y}$

$\Rightarrow x = ab, \qquad y = a - ab = a(1-b)$

Jacobian of transformation J is given by

$$J = \frac{\partial(x,y)}{\partial(a,b)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} \end{vmatrix} = \begin{vmatrix} b & 1-b \\ a & -a \end{vmatrix} = -a$$

As X and Y range form 0 to ∞

$\Rightarrow$ a ranges from 0 to ∞ and b from 0 to 1

Hence joint distribution A and B is

$$dG(a,b) = g(a,b)\,dadb = \frac{1}{\overline{|(u)}\overline{|(v)}}e^{-a}(ab)^{u-1}(a(1-b))^{v-1}|J|\,dadb$$

$$= \frac{1}{\Gamma(u)\Gamma(v)} e^{-a} a^{u+v-1} b^{u-1} (1-b)^{v-1}\, da\,db$$

$$= \left(\frac{e^{-a} a^{u+v-1} da}{\Gamma(u+v)} \right) \left(\frac{1}{\beta(u,v)} b^{u-1} (1-b)^{v-1}\, db \right)$$

$$= \left[g_1(a)\,da \right] \left[g_2(b)\,db \right]$$

where $g_1(a) = \frac{1}{\Gamma(u+v)} e^{-a} a^{u+v-1} \;;\quad 0 < a < \infty$

$$g_2(b) = \frac{1}{\beta(u,v)} b^{u-1} (1-b)^{v-1} \;;\quad 0 < b < 1$$

$\Rightarrow X+Y$ and $\frac{X}{X+Y}$ are independently distributed with $X+Y$ as

$\gamma(u+v)$ variate and $\frac{X}{X+Y}$ as $\beta(u,v)$ variate

18. Find the mean deviation from the mean for a normal distribution.

(September 2009, April 2009)

Sol. M.D. (about mean) = $\int_{-\infty}^{\infty} |x-\mu| f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu| e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{\frac{-z^2}{2}}\,dz \,. \qquad \left(\frac{x-\mu}{\sigma} = z \right)$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} |z| e^{\frac{-z^2}{2}}\,dz$$

$\left(\because \text{ the integral } |z| e^{\frac{-z^2}{2}} \text{ is an even function of z.} \right)$

Since in $[0,\infty]$, $|z| = z$, we have

M.D. (about mean) $= \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} z e^{\frac{-z^2}{2}}\,dz$

$$= \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t}\,dt \qquad \left[taking\ \frac{z^2}{2} = t \right]$$

$$= \sigma\sqrt{\frac{2}{\pi}} \left| \frac{e^{-t}}{-1} \right|_0^{\infty} = \sqrt{\frac{2}{\pi}}\sigma = \frac{4}{5}\sigma \text{ (approx.)}$$

19. If X is a uniformly distributed random variable with mean 1 and variance $\frac{4}{3}$, find $P(X<0)$.

(April 2009)

Sol. Let X ~ U $[a,b]$, so that $p(x) = \frac{1}{b-a}, a < x < b$.

We are given:

Mean = $\frac{1}{2}(b+a) = 1 \qquad \Rightarrow b+a = 2$

and Var(X) = $\frac{1}{12}(b-a)^2 = \frac{4}{3} \qquad \Rightarrow b-a = \pm 4$.

Solving, we get $a = -1$ and $b = 3$; $(a<b)$

Thus, $P(X<0) = \int_{-1}^{0} p(x)\,dx = \frac{1}{4}|x|_{-1}^{0} = \frac{1}{4}$

$\therefore\ p(x) = \frac{1}{4}; -1 < x < 3$.

6

BIVARIATE RANDOM VARIABLES

1. **Define Joint probability density function of a pair of continuous random variables. (September 2013)**

Sol. The joint probability density function for the continuous random variables X and Y is defined by

(a) $f(x,y) \geq 0$

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$

2. **If the joint density function of two random variables X and Y is given by:**

$$f(x)=\begin{cases}\frac{2}{3}(x+2y) & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{elsewhere}\end{cases}$$ **(September 2013)**

Find the conditional mean and the conditional variance of X, given $Y=\frac{1}{2}$.

Sol. Given $f(x)=\begin{cases}\frac{2}{3}(x+2y) & ;0<x<1,\ 0<y<1 \\ 0 & ;\text{ elsewhere}\end{cases}$

$$\therefore f_Y\left(y=\frac{1}{2}\right)=\int_{x=0}^{1}\frac{2}{3}(x+1)dx$$

$$=\frac{2}{3}\left[\frac{x^2}{2}+x\right]_0^1=\frac{2}{3}\left[\frac{1}{2}+1\right]=\frac{2}{3}\left(\frac{3}{2}\right)=1$$

$$E\left[X\mid Y=\frac{1}{2}\right]=\frac{\int_0^1\frac{2}{3}x(x+1)dx}{f_Y\left(y=\frac{1}{2}\right)}=\int_0^1\frac{2}{3}x(x+1)dx$$

$$=\frac{2}{3}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1=\frac{2}{3}\left[\frac{1}{3}+\frac{1}{2}\right]$$

$$=\frac{5}{9}$$

$\therefore$ Conditional mean of X given Y = $\frac{1}{2}$ is $\frac{5}{9}$

Now conditional variance of X given Y is given by

$$V(X\mid Y=y)=E\left[\{X-E(X\mid Y=y)\}^2\mid Y=y\right]$$

$$\therefore V\left(X\mid Y=\frac{1}{2}\right)=E\left[\left\{X-E\left(X\mid Y=\frac{1}{2}\right)\right\}^2\mid Y=\frac{1}{2}\right]$$

$$=\frac{\int_{x=0}^{1}\left(x-\frac{5}{9}\right)^2\left(\frac{2}{3}(x+1)\right)dx}{f_Y\left(y=\frac{1}{2}\right)}$$

$$=\frac{2}{3}\int_0^1\left(x-\frac{5}{9}\right)^2(x+1)dx$$

$$=\frac{2}{3}\int_0^1\left(x^2-\frac{10x}{9}+\frac{25}{81}\right)(x+1)dx$$

$$=\frac{2}{3}\int_0^1\left(x^3-\frac{x^2}{9}-\frac{65x}{81}+\frac{25}{81}\right)dx$$

$$=\frac{2}{3}\left[\frac{x^4}{4}-\frac{x^3}{27}-\frac{65x^2}{162}+\frac{25}{81}x\right]_0^1$$

$$=\frac{2}{3}\left[\frac{1}{4}-\frac{1}{27}-\frac{65}{162}+\frac{25}{81}\right]$$

$$=\frac{2}{3}\left(\frac{39}{324}\right)=\frac{13}{162}$$

3. **Let $M(t_1,t_2)$ be the moment generating function of two random variables X and Y. Shwo that X and Y are independent iff: (September 2013, September 2011)**
$M(t_1, t_2)=M(t_1, 0)M(0, t_2)$.

Sol. Let X and Y be independent. Then

$$M(t_1,t_2)=E\left(e^{t_1X+t_2Y}\right)=E\left(e^{t_1X}e^{t_2Y}\right)=E\left(e^{t_1X}\right)E\left(e^{t_2Y}\right)$$

$=M(t_1,0)M(0,t_2)$

Conversely, if

$M(t_1,t_2)=M(t_1,0)M(0,t_2)$

$\iint e^{t_1x+t_2y}f(x,y)\,dxdy=\left[\int e^{t_1x}f_1(x)\,dx\right]\left[\int e^{t_2x}f_2(y)\,dy\right]$

$=\iint e^{t_1x+t_2y}f_1(x)f_2(y)\,dxdy$

By uniqueness of MGF

$f(x,y)=f_1(x)f_2(y)$

$\Rightarrow$ X and Y are independent

4. **If the random variable X and Y have joint density function given by:**

$$f(x,y)=\begin{cases}\dfrac{1}{210}(2x+y) & ;\ 2<x<6, 0<y<5\\ 0 & ;\ \text{otherwise}\end{cases}$$

Find (i) The Marginal density function for x **(April 2013)**

(ii) $P(X+Y>4)$.

Sol. (i) Marginal density of x

$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,dy=\frac{1}{210}\int_0^5(2x+y)\,dy$

$=\frac{1}{210}\left[2xy+\frac{y^2}{2}\right]_0^5$

$=\frac{1}{210}\left[10x+\frac{25}{2}\right]=\frac{x}{21}+\frac{5}{84}$

(ii) $P(X+Y>1)=1-P(X+Y<4)$

$=1-\frac{1}{210}\int_2^4\int_0^{4-x}(2x+y)\,dydx$

$=1-\frac{1}{210}\int_2^4\left[2xy+\frac{y^2}{2}\right]_0^{4-x}dx$

$=1-\frac{1}{210}\int_2^4\left(-\frac{3x^2}{2}+4x+8\right)dx$

$=1-\frac{1}{210}\left[-\frac{x^3}{2}+2x^2+8x\right]_2^4$

$=1-\frac{1}{210}\left[(32)-(20)\right]=1-\frac{12}{210}$

$=\frac{33}{35}$

5. **If coefficient of correlation between two jointly normally distributed variable is zero, then show that they are independent.** **(April 2013)**

Sol. The joint prob. Distribution function of X and Y is given by

$$f(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}}\times\exp\left[-\frac{1}{2(1-r^2)}\left\{\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right.\right.$$

$$\left.\left.-2r\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)+\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right\}\right]$$

If $r=0$, then

$$f(x,y)=\frac{1}{\sigma_X\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right]$$

$$\times\frac{1}{\sigma_Y\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{X-\mu_Y}{\sigma_Y}\right)^2\right]$$

$=f_1(x)f_2(y)$

which are the prob. distribution function of X and Y separately.

Hence X and Y are independent.

6. **If $f(x,y)=2-x-y,(0,\le x\le 1, 0\le y\le 1)$ *and* 0 elsewhere, find:**

(i) The marginal probability functions

(ii) The conditional probability functions. **(September 2012)**

Sol. (i) Marginal p.d.f

$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,dy=\int_0^1(2-x-y)\,dy$

$=\left[2y-xy-\frac{y^2}{2}\right]_0^1$

$=2-x-\frac{1}{2}=\frac{3}{2}-x$

$$f_X(x)=\frac{3}{2}-x \quad \text{for} \quad 0\le x\le 1$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1 (2-x-y)\,dx$$

$$=\left[2x-\frac{x^2}{2}-xy\right]_0^1$$

$$=2-\frac{1}{2}-y=\frac{3}{2}-y$$

$$f_Y(y)=\frac{3}{2}-y \quad \text{for} \quad 0\le y\le 1$$

(ii) Conditional p.d.f

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{(2-x-y)}{\frac{3}{2}-y}$$

$$=\frac{2(2-x-y)}{(3-2y)},0\le x\le 1$$

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{(2-x-y)}{\frac{3}{2}-x}$$

$$=\frac{(2-x-y)}{(3-2x)},0\le y\le 1$$

7. **The joint density function of two continuous random variables X and Y is:**

$$f(x,y)=\begin{cases} cxy, & 0<x<4, 1<y<5 \\ 0, & \text{otherwise}\end{cases}$$ **(September 2012)**

(i) Find the value of c

(ii) Find $P(1<x<2, 2<y<3)$

(iii) Find $P(x\ge 3, y\le 2)$.

Sol. (a) Since the total probability is equal to 1, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

thus we have

$$\int_{x=0}^{4}\int_{y=1}^{5} C\,x\,y\,dx\,dy=C\int_{x=0}^{4}\left[\int_{y=1}^{5} x\,y\,dy\right]dx$$

$$=C\int_{x=0}^{4}\left.\frac{xy^2}{2}\right|_{y=1}^{5} dx$$

$$=C\int_0^4\left(\frac{25}{2}x-\frac{x}{2}\right)dx$$

$$=C\int_0^4 12x\,dx=6Cx^2\Big|_0^4=96C$$

$\therefore$ 96 C = 1

$\Rightarrow$ C = $\frac{1}{96}$.

The joint density function using the value of C thus becomes

$$f(x,y)=\begin{cases} \frac{xy}{96} &, \quad 0<x<4,\ 1<y<5 \\ 0 &, \quad \textit{Otherwise}\end{cases}$$

(b) $P(1<X<2, 2<Y<3)=\int_{x=1}^{2}\int_{y=2}^{3}\frac{xy}{96}dx\,dy$

$$=\frac{1}{96}\int_{x=1}^{2}\int_{y=2}^{3} x\,y\,dy\,dx=\frac{1}{96}\int_{x=1}^{2}\left.\frac{xy^2}{2}\right|_{y=2}^{3} dx$$

$$=\frac{1}{96}\int_{x=1}^{2}\frac{5x}{2}dx=\frac{5}{192}\left(\frac{x^2}{2}\right)\Big|_1^2=\frac{5}{128}$$

(c) $P(X\ge 3, Y\le 2)=\int_{x=3}^{4}\int_{y=1}^{2}\frac{xy}{96}dx\,dy$

$$=\frac{1}{96}\int_{x=3}^{4}\left[\int_{y=1}^{2} x\,y\,dy\right]dx$$

$$=\frac{1}{96}\int_{x=3}^{4}\left.\frac{xy^2}{2}\right|_{y=1}^{2} dx$$

$$=\frac{1}{96}\int_3^4\frac{3x}{2}dx=\frac{7}{128}$$

8. The random variable X and Y have the joint probability density function

$$f(x,y)=\begin{cases} x+y & 0<x<1, \quad 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the coefficient of correlation of X and Y. **(April 2012)**

Sol. $E(X)=\int_0^1\int_0^1 x(x+y)\,dxdy=\frac{7}{12}$

$$E(Y)=\int_0^1\int_0^1 y(x+y)\,dxdy=\frac{7}{12}$$

$$E(XY)=\int_0^1\int_0^1 xy(x+y)\,dxdy=\frac{1}{3}$$

$$E(X^2)=\int_0^1\int_0^1 x^2(x+y)\,dxdy=\frac{5}{12}$$

$$E(Y^2)=\int_0^1\int_0^1 y^2(x+y)\,dxdy=\frac{5}{12}$$

$$\sigma_X^2=Var(X)=E(X^2)-(E(X))^2$$

$$=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}$$

$$\sigma_Y^2=Var(Y)=E(Y^2)-(E(Y))^2$$

$$=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}$$

$$\sigma_{XY}=Cov(X,Y)=E(XY)-E(X)E(Y)$$

$$=\frac{1}{3}-\left(\frac{7}{12}\right)\left(\frac{7}{12}\right)=\frac{-1}{144}$$

Coefficient of correlation

$$\rho=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}=\frac{\frac{-1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}}=\frac{-1}{11}$$

$$=-0.0909(approx)$$

9. Let $f(x_1|x_2)=\begin{cases}\frac{c_1x_1}{x_2^2}, 0<x_1<x_2, 0<x_2<1 \\ 0 \quad \text{elsewhere}\end{cases}$ **and** $f_2(x_2)=\begin{cases}c_2x_2^4, 0<x_2<1 \\ 0 \quad \text{elsewhere}\end{cases}$

(September 2011)

Find (i) constants c_1 and c_2 (ii) joint p.d.f. of x_1 and x_2

(iii) $P\left(\frac{1}{4}<X_1<\frac{1}{2}\middle|X_2=\frac{5}{8}\right)$ (iv) $P\left(\frac{1}{4}<X_1<\frac{1}{2}\right)$

Sol. $f(x_1|x_2)=\frac{f(x_1,x_2)}{f_2(x_2)}$

$$\frac{c_1x_1}{x_2^2}=\frac{f(x_1,x_2)}{c_2x_2} \Rightarrow f(x_1,x_2)=c_1c_2x_1x_2^2$$

(i) Joint p.d.f. of x_1 and x_2

$$f(x_1,x_2)=\begin{cases}c_1c_2x_1x_2^2, & 0<x_1<x_2, \quad 0<x_2<1 \\ 0 & \text{elsewhere}\end{cases}$$

(ii) $\int_0^1 f_2(x_2)\,dx_2=1 \qquad \Rightarrow \int_0^1 c_2x_2^4\,dx_2=1$

$$\Rightarrow c_2\left[\frac{x_2^5}{5}\right]_0^1=1 \Rightarrow c_2\left[\frac{1}{5}\right]=1 \quad \Rightarrow c_2=5$$

Also $\int_0^1\int_0^{x_2} f(x_1,x_2)\,dx_1dx_2=1$

$$\Rightarrow c_1c_2\int_0^1\int_0^{x_2} x_1x_2^2\,dx_1dx_2=1$$

$$\Rightarrow 5c_1\int_0^1\left[\frac{x_1^2}{2}\right]_0^{x_2} x_2^2\,dx_2=1 \Rightarrow 5c_1\int_0^1\left[\frac{x_2^2}{2}\right]x_2^2\,dx_2=1$$

$$\Rightarrow \frac{5c_1}{2}\int_0^1 x_2^4\,dx_2=1 \Rightarrow \frac{5c_1}{2}\left[\frac{x_2^5}{2}\right]_0^1=1$$

$$\Rightarrow \frac{5c_1}{2}\left[\frac{1}{5}\right]=1 \quad \Rightarrow c_1=2$$

(iii) $P\left(\frac{1}{4}<X_1<\frac{1}{2}\middle|X_2=\frac{5}{8}\right)=\frac{P\left(\frac{1}{4}<X_1<\frac{1}{2}\cap X_2=\frac{5}{8}\right)}{P\left(X_2=\frac{5}{8}\right)}$

$$= \frac{P\left(\frac{1}{4} < X_1 < \frac{1}{2} \cap X_2 = \frac{5}{8}\right)}{P\left(X_1 < \frac{5}{8} \cap X_2 = \frac{5}{8}\right)}$$

Now $\quad f\left(x_1 \mid x_2 = \frac{5}{8}\right) = \frac{2x_1}{\left(\frac{5}{8}\right)^2} = \frac{128x_1}{25}$

$$\Rightarrow P\left(\frac{1}{4} < X_1 < \frac{1}{2} \middle| X_2 = \frac{5}{8}\right) = \frac{\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{128x_1}{25} dx_1}{\int_0^{\frac{5}{8}} \frac{128x_1}{25} dx_1} = \frac{\int_{\frac{1}{4}}^{\frac{1}{2}} x_1 dx_1}{\int_0^{\frac{5}{8}} x_1 dx_1}$$

$$= \frac{\left[x_1^2\right]_{1/4}^{1/2}}{\left[x_1^2\right]_0^{5/8}} = \frac{\left[\frac{1}{4} - \frac{1}{16}\right]}{\left[\frac{25}{64} - 0\right]} = \frac{3}{16} \cdot \frac{64}{25} = \frac{12}{25}$$

(iv) $P\left(\frac{1}{4} < X_1 < \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_0^1 f(x_1, x_2)\, dx_2 dx_1$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \int_0^1 10x_1 x_2^2\, dx_2 dx_1$$

$$= 10\left[\frac{x_1^2}{2}\right]_{1/4}^{1/2} \left[\frac{x_2^3}{3}\right]_0^1$$

$$= \frac{10}{2}\left[\frac{1}{4} - \frac{1}{16}\right]\left[\frac{1}{3} - 0\right]$$

$$= 5\left(\frac{3}{16}\right)\left(\frac{1}{3}\right) = \frac{5}{16}$$

11. Write a short note on: (a) Joint distribution
(b) Correlation coefficient **(April 2011)**

Sol. **(a) Joint distribution:** Let (x, y) be a two dimensional random variable then their joint distribution function is denoted by $F_{XY}(x, y)$

$$F_{XY}(x, y) = P(-\infty < X \le x, \quad -\infty < Y \le y)$$

Also the properties

(i) for real numbers a_1, b_1, a_2 and b_2

$$p(a_1 < X \le b_1,\ a_2 < Y \le b_2) = F_{XY}(b_1, b_2) + F_{XY}(a_1, a_2) - F_{XY}(a_1, b_2) - F_{XY}(b_1, a_2)$$

(ii) $F(-\infty, y) = 0 = F(x, +\infty)$

$F(+\infty, +\infty) = 1$

(iii) $\frac{\partial^2}{\partial x \partial y} F = f(x, y)$

(c) Correlation coefficient: Correlation coefficient between two random variables X and Y usually by $r(X, Y)$ is a numerical measure of linear relationship between them and is defined as

$$r(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

where $\quad Cov(X, Y) = E\left[(X - E(X))(Y - E(Y))\right]$

$$\sigma_X^2 = E\left[(X - E(X))^2\right]$$

$$\sigma_Y^2 = E\left[(Y - E(Y))^2\right]$$

Also $-1 \le r(X, Y) \le 1$

12. Suppose that the joint p.d.f. of two-dimensional random variable (X, Y) is given by: **(September 2010)**

$$f(x, y) = \begin{cases} x^2 + \frac{xy}{3} & 0 < x < 1,\ 0 < y < 2 \\ 0 & elsewhere \end{cases}$$

Compare (i) $P\left(X > \frac{1}{2}\right)$ **(ii)** $P\left(Y < \frac{1}{2} \middle| X < \frac{1}{2}\right)$.

Sol. (i) $P\left(X > \frac{1}{2}\right) = \int_{1/2}^{1} \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy dx$

$$= \int_{1/2}^{1} \left[x^2 y + \frac{xy^2}{6}\right]_0^2 dx$$

$$= \int_{1/2}^{1} \left(2x^2 + \frac{2x}{3}\right) dx$$

$$=\left[\frac{2x^3}{3}+\frac{x^2}{3}\right]_{1/2}^{1}$$

$$=\left(\frac{2}{3}+\frac{1}{3}\right)-\left(\frac{1}{12}+\frac{1}{12}\right)=1-\frac{1}{6}=\frac{5}{6}$$

(ii) $P\left(Y<\frac{1}{2}\middle|X<\frac{1}{2}\right)=\frac{P\left(Y<\frac{1}{2}\cap X<\frac{1}{2}\right)}{P\left(X<\frac{1}{2}\right)}$

$$=\frac{\int_0^{1/2}\int_0^{1/2}\left(x^2+\frac{xy}{3}\right)dydx}{1-P\left(X>\frac{1}{2}\right)}$$

$$=\frac{\int_0^{1/2}\left[x^2y+\frac{xy^2}{6}\right]_0^{1/2}dx}{1-\frac{5}{6}}$$

$$=\frac{\int_0^{1/2}\left[\frac{x^2}{2}+\frac{x}{24}\right]dx}{\frac{1}{6}}=6\left[\frac{x^3}{6}+\frac{x^2}{48}\right]_0^{1/2}$$

$$=\left[x^3+\frac{x^2}{8}\right]_0^{1/2}$$

$$=\frac{1}{8}+\frac{1}{32}=\frac{5}{32}$$

13. **If X and Y are standardized random variables and $r(aX+bY,bX+aY)=\frac{1+2ab}{a^2+b^2}$, find $r(x,y)$, the coefficient of correlation between X and Y. (September 2010)**

Sol. Since X and Y are standardized random variables

$\Rightarrow E(X)=E(Y)=0$

And $Var(X)=Var(Y)=1 \quad \Rightarrow E(X^2)=E(Y^2)=1$

and $Cov(X,Y)=E(XY)$

$\Rightarrow E(XY)=r(X,Y)\sigma_X\sigma_Y=r(X,Y)$

Now

$r(aX+bY,bX+aY)$

$$=\frac{E\left[(aX+bY)(bX+aY)\right]-E(aX+bY)E(bX+aY)}{\left[var(aX+bY)Var(bX+aY)\right]^{1/2}}$$

$$=\frac{E\left[abX^2+a^2XY+b^2YX+abY^2\right]-0}{\left(\left[a^2\,var(X)+b^2\,var(Y)+2abCov(X,Y)\right]\left[b^2\,var(X)+a^2\,var(Y)+2abCov(X,Y)\right]\right)^{1/2}}$$

$$=\frac{2ab+(a^2+b^2)r(X,Y)}{a^2+b^2+2abr(X,Y)}$$

$$\Rightarrow \frac{1+2ab}{a^2+b^2}=\frac{(a^2+b^2)r(X,Y)+2ab}{(a^2+b^2)+2ab.r(X,Y)}$$

$$\Rightarrow (a^2+b^2)(1+2ab)+2ab.r(X,Y)(1+2ab)$$

$$=(a^2+b^2)^2r(X,Y)+2ab(a^2+b^2)$$

$$\Rightarrow (a^4+b^4+2a^2b^2-2ab-4a^2b^2).r(X,Y)=a^2+b^2$$

$$\Rightarrow \left[(a^2-b^2)^2-2ab\right].r(X,Y)=a^2+b^2$$

$$\Rightarrow r(X,Y)=\frac{a^2+b^2}{(a^2-b^2)^2-2ab}$$

14. **The joint density function of the two dimensional random variable (X, Y) is given by:** $f(x,y)=\begin{cases}\frac{8}{9}xy & ; \quad 1\le x\le y\le 2\\ 0 & \quad elsewhere\end{cases}$ **(April 2010)**

Find the conditional density function of Y given X = x and conditional density function of X given Y = y.

Sol. $f_X(x)=\int_x^2 f(x,y)dy=\int_x^2\frac{8}{9}xydy$

$$=\frac{8}{9}x\left[\frac{y^2}{2}\right]_x^2=\frac{4}{9}x(4-x^2)$$

$$f_X(x)=\begin{cases}\frac{4}{9}x(4-x^2); & 1\le x\le 2\\ 0 & ; \quad \text{otherwise}\end{cases}$$

$$f_Y(y)=\int_1^y f(x,y)\,dx=\int_1^y \frac{8}{9}xy\,dx=\frac{8}{9}y\left[\frac{x^2}{2}\right]_1^y=\frac{4}{9}y\left(y^2-1\right)$$

$$f_Y(y)=\frac{4}{9}y\left(y^2-1\right) \quad ; \quad 1\le y\le 2$$

Conditional density function of Y given X = x

$$F_{Y|X}(y\,|\,x)=\frac{f(x,y)}{f_X(x)}=\frac{\frac{8}{9}xy}{\frac{4}{9}x\left(4-x^2\right)}=\frac{2y}{4-x^2} \quad ; \quad x\le y\le 2$$

Conditional density function of X given Y = y

$$F_{X|Y}(x\,|\,y)=\frac{f(x,y)}{f_Y(x)}=\frac{\frac{8}{9}xy}{\frac{4}{9}y\left(y^2-1\right)}=\frac{2x}{y^2-1} \quad ; \quad 1\le x\le y$$

15. The random variable X and Y have the joint distribution given by the probability density function:

$$f(x,y)=\begin{cases}6(1-x-y) & \text{for } x>0, y>0, x+y<1\\ 0 & \text{elsewhere}\end{cases}$$

(April 2010)

Examine if X and Y are independent.

Sol. $f_X(x)=\int_0^{1-x} 6(1-x-y)\,dy$

$=\left[6y-6xy-3y^2\right]_0^{1-x}$

$=3(1-x)^2$

$f_Y(y)=\int_0^{1-x} 6(1-x-y)\,dy$

$=\left[6x-3x^2-6xy\right]_0^{1-y}$

$=3(1-y)^2$

$\Rightarrow f_{XY}(x,y)\ne f_X(x)\,f_Y(y)$

$\Rightarrow$ X and Y are not independent

16. Show that correlation coefficient is independent of change of origin and scale. **(April 2010)**

Sol. Let $U=\dfrac{X-a}{h}$, $\quad V=\dfrac{Y-b}{k} \quad \Rightarrow X=a+hU\,,\quad Y=b+kV$

where a, b, h, k are constant, $h>0$, $k>0$.

We have changed the origin to (a, b) and changed the scale by $\frac{1}{h}$ and $\frac{1}{k}$

We will prove $\quad r(X,Y)=r(U,V)$

$X=a+hU \quad$ and $\quad Y=b+kV$

$E(X)=a+hE(U)$ and $E(Y)=b+kE(V)$

$\therefore\ X-E(X)=h\left[U-E(U)\right]$ and $Y-E(Y)=k\left[V-E(V)\right]$

$\Rightarrow \operatorname{cov}(X,Y)=E\left[\left(X-E(X)\right)\left(Y-E(Y)\right)\right]$

$=E\left[h\left(U-E(U)\right)k\left(V-E(V)\right)\right]$

$=hk\,E\left[\left(U-E(U)\right)\left(V-E(V)\right)\right]$

$=hk\,\operatorname{cov}(U,V)$

$\sigma_X^{\,2}=E\left[\left(X-E(X)\right)^2\right]=E\left[h^2\left(U-E(U)\right)^2\right]=h^2\sigma_U^{\,2}$

$\Rightarrow \sigma_X=h\sigma_U, \quad h>0$

Similarly $\sigma_Y^{\,2}=k^2\sigma_V^{\,2}$

$\Rightarrow \sigma_Y=k\sigma_V, \quad k>0$

$$\Rightarrow r(X,Y)=\frac{\operatorname{cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{hk\,\operatorname{cov}(U,V)}{hk\sigma_U\sigma_V}=\frac{\operatorname{cov}(U,V)}{\sigma_U\sigma_V}=r(U,V)$$

$\Rightarrow r(X,Y)=r(U,V)$

17. The joint probability density function of a two-dimensional random variable (x,y) **is given by** $f(x,y)=\begin{cases}2 \ ; & 0<x<1,\ 0<y<x\\ 0 \ ; & \text{elsewhere}\end{cases}$

(i) Find the marginal density functions of X and Y

(ii) Find the conditional density function of Y, given X = x **and conditional density function of X, given Y = y**

(iii) Check the independence of X and Y. **(September 2009)**

Sol. (a) The marginal prob. distribution function of X is

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^x 2\,dy=2x \ \text{ for } 0<x<1,$$

and for other values of x, it is $f_X(x)=0$.

The marginal prob. distribution function of Y is

$$g_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_y^1 2dx=2(1-y) \text{ for } 0<y<1.$$

and for other values of y, it is $g_Y(y)=0$

(b) The conditional density function by definition of Y given X, $(0<x<1)$ is

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{2}{2x}=\frac{1}{x}, 0<y<x.$$

Similarly, the conditional density function of X given Y, $(0<y<1)$ is

$$f_{X|Y}(x|y)=\frac{f(x,y)}{g_Y(y)}=\frac{2}{2(1-y)}=\frac{1}{1-y}, y<x<1.$$

(c) Since $f_X(x)g_Y(y)=2(2x)(1-y)\neq f(x,y)$, the variables X and Y are not independent.

18. The random variables X and Y are jointly normally distributed. Let

$U = X\cos\alpha + Y\sin\alpha,\quad V = Y\cos\alpha - X\sin\alpha$

Show that X and Y will be uncorrelated If $\tan 2\alpha=\dfrac{2r\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2}$

where r = correlation coefficient between X and Y

$\sigma_X^2=\text{var}(X)$ **and** $\sigma_Y^2=\text{var}(Y)$. **(September 2009, April 2009)**

Sol. We have $U=X\cos\alpha+Y\sin\alpha$, $V=Y\cos\alpha-X\sin\alpha$

$\Rightarrow E(U)=E(X)\cos\alpha+E(Y)\sin\alpha, E(V)=E(Y)\cos\alpha-E(X)\sin\alpha$

finding U – E(U) and V – E(V) and putting in Cov (U, V), we get

$Cov(U,V)=E\left[\{U-E(U)\}\{V-E(V)\}\right]$

$=E\left[\left[\{X-E(X)\}\cos\alpha+\{Y-E(Y)\}\sin\alpha\right]\right.$

$\left.\left[\{Y-E(Y)\}\cos\alpha-\{X-E(X)\}\sin\alpha\right]\right]$

$=\cos^2\alpha.Cov(X,Y)-\sin\alpha.\cos\alpha.\sigma_X^2+\sin\alpha.\cos\alpha.\sigma_Y^2-\sin^2\alpha Cov(X,Y)$

$=(\cos^2\alpha-\sin^2\alpha)Cov(X,Y)-\sin\alpha\cos\alpha(\sigma_X^2-\sigma_Y^2)$

$=\cos 2\alpha Cov(X,Y)-\sin\alpha\cos\alpha(\sigma_X^2-\sigma_Y^2)$

U and V will be uncorrelated if and only if

$r(U,V)=0$, *i.e.*, If $\text{Cov}(U,V)=0$

i.e., if $\cos 2\alpha\ Cov(X,Y)-\sin\alpha\cos\alpha(\sigma_X^2-\sigma_Y^2)=0$

or if $\cos 2\alpha.r\sigma_X\sigma_Y=\dfrac{\sin 2\alpha}{2}(\sigma_X^2-\sigma_Y^2)$

or if $\tan 2\alpha=\dfrac{2r\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2}$

19. The joint probability density function of bivariate random variables (X, Y) is given by: $f(x,y)=4xye^{(x^2+y^2)}; x>0, y>0$

(i) Test whether X and Y are independent.

(ii) Find conditional density of X given that Y = y. **(April 2009)**

Sol. The marginal density of X is given by

$$f_X(x)=\int_{y=0}^{\infty} f(x,y)\,dy=\int_{y=0}^{\infty} 4xye^{-(x^2+y^2)}dy$$

$$=4xe^{-x^2}\int_{y=0}^{\infty} y.e^{-y^2}dy=4x.e^{-x^2}\int_0^{\infty} e^{-t}\frac{dt}{2}$$

$$=2x.e^{-x^2}\left[-e^{-t}\right]_0^{\infty}=2xe^{-x^2},\quad x\geq 0$$

Similarly, the marginal density of Y is given by

$$g_Y(y)=\int_{x=0}^{\infty} f(x,y)\,dx=\int_{x=0}^{\infty} 4xye^{-(x^2+y^2)}dx$$

$=2ye^{-y^2}, y\geq 0.$

(a) Since $f_X(x).g_Y(y)=\left(2xe^{-x^2}\right)\left(2ye^{-y^2}\right); x,y\geq 0.$

$=4xye^{-(x^2+y^2)}; x\geq 0,\ y\geq 0$

$=f(x,y)$

$\therefore$ X and Y are independent of each other.

$$f_{X|Y}(X|Y)=\frac{f(x,y)}{g_Y(y)}=\frac{4xye^{-(x^2+y^2)}}{2ye^{-y^2}}$$

$=2xe^{-x^2};\ x\geq 0$

1

ROOTS OF EQUATION

1. Using Newton Raphson method and by performing 4 iterations find a root of the equation $\cos x = 3x - 1$. (September 2013)

Sol. Let $f(x) = \cos x - 3x + 1$

$f'(x) = -\sin x + 3$

Since $f(0) = 1 - 0 + 1 = 2 = +ve$

and $f\left(\frac{\pi}{4}\right) = 0.707 - \frac{3\pi}{4} + 1 = -0.649 = -ve$

$\therefore$ The root lies between 0 and $\frac{\pi}{4}$

Take the initial value $x_0 = \frac{\pi}{4}$

By Newton-Raphson method, *i*th iteration is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$\therefore$ **The first approximation is**

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= \frac{\pi}{4} - \frac{\left(\cos\frac{\pi}{4} - 3\frac{\pi}{4} + 1\right)}{-\sin\frac{\pi}{4} + 3} = 0.6104$$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6104 - \frac{[\cos(0.6104) - 3\times 0.6104 + 1]}{-\sin(0.6104) - 3} = 0.6071$$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6071 - \frac{[\cos(0.6071) - 3\times 0.6071 + 1]}{-\sin(0.6071) - 3} = 0.6071$$

Since the second and third approximations are same

$\therefore$ the required root is 0.6071 correct to four decimal places.

2. Find the order of convergence of Newton-Raphson's method of solving a non linear equation. (September 2013)

Sol. Newton Raphson iterative formula is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Let α be the root of the equation $f(x) = 0$ and e_i and e_{i+1} are the errors *i*th and $(i+1)$ the iterations

Then $x_i - \alpha = e_i$ and $x_{i+1} - \alpha = e_{i+1}$

Or $x_i = \alpha + e_i$ and $x_{i+1} = \alpha + e_{i+1}$

Putting these values in (1), we obtain

$$\alpha + e_{i+1} = \alpha + e_i - \frac{f(\alpha + e_i)}{f'(\alpha + e_i)}$$

$$\Rightarrow e_{i+1} = e_i - \frac{f(\alpha + e_i)}{f'(\alpha + e_i)} \quad (2)$$

Expanding $f(\alpha + e_i)$ and $f'(\alpha + e_i)$ by Taylor's Theorem

$$f(\alpha + e_i) = f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2} f''(\alpha) + ...$$

$$= e_i\left[f'(\alpha) + \frac{e_i}{2} f''(\alpha)\right] \quad (3)$$

($\because$ α is root of $f(x) = 0$ $\therefore$ $f(\alpha) = 0$ and e_i is Small neglecting higher order terms of e_i)

and $f'(\alpha + e_i) = f'(\alpha) + e_i f''(\alpha) + ...$

$$= f'(\alpha) + e_i f''(\alpha) \qquad (4)$$

From (2), (3) and (4), we get

$$e_{i+1} = e_i - \frac{e_i\left[f'(\alpha) + \frac{e_i}{2} f''(\alpha)\right]}{\left[f'(\alpha) + e_i f''(\alpha)\right]}$$

$$= \frac{e_i\left[f'(\alpha) + e_i f''(\alpha)\right] - e_i\left[f'(\alpha) + \frac{e_i}{2} f''(\alpha)\right]}{f'(\alpha) + e_i f''(\alpha)}$$

$$= \frac{e_i^2 f''(\alpha)}{2\left[f'(\alpha) + e_i f''(\alpha)\right]} = \frac{e_i^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)\left[1 + e_i \frac{f''(\alpha)}{f'(\alpha)}\right]}$$

$$= \frac{e_i^2}{2} \frac{f''(\alpha)}{f'(\alpha)}\left[1 + e_i \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$= \frac{e_i^2 f''(\alpha)}{2 f'(\alpha)}\left[1 - \frac{e_i f''(\alpha)}{f'(\alpha)} + \ldots\right] \qquad \text{(By Binomial expansion)}$$

$$= \frac{e_i^2 f''(\alpha)}{2 f'(\alpha)} \qquad \text{(Neglecting } e_i^3 \text{ and its higher order terms)}$$

Take $\frac{f''(\alpha)}{2f'(\alpha)} = m$ where m is finite

We have $e_{i+1} = m e_i^2$

$$\Rightarrow \frac{e_{i+1}}{(e_i)^2} = m$$

Since index of e_i is 2

$\therefore$ order of convergence is 2.

Hence Newton-Raphson method has a quadratic convergence.

3. **Apply Regula Falsi method with 4 iterations to find a root of the equation $x^3 - x^2 - 2 = 0$ that lies in (1, 2). (April 2013)**

Sol. $f(x) = x^3 - x^2 - 2 = 0 \quad x_0 = 1 \quad x_1 = 2$

Using Regula falsi Method $x_{k+1} = \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$

I^st Iteration

$f(x_0) = (1)^3 - (1)^2 - 2 = -2$

$f(x_1) = (2)^3 - (2)^2 - 2 = 2$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1(2) - 2(-2)}{2 - (-2)} = \frac{6}{4} = 1.5$$

$f(1.5) = (1.5)^3 - (1.5)^2 - 2 = -0.875$

$f(x_1) f(x_2) < 0$ Substitute $x_0 = x_2, \ x_1 = x_1$

II^nd Iteration

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.5)(2) - 2(-0.875)}{2 - (-0.875)} = 1.6521$$

$f(1.6521) = (1.6521)^3 - (1.6521)^2 - 2 = -0.22$

$f(x_2) f(x_1) < 0$ Substitute $x_0 = x_2 \ x_1 = x_1$

III^rd Iteration

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.6521)(2) - 2(-0.22)}{2 - (-0.22)} = 1.6865$$

$f(1.6865) = (1.6865)^3 - (1.6865)^2 - 2 = -0.0474$

$f(x_2) f(x_1) < 0$ Substitute $x_0 = x_2 \ x_1 = x_1$

IV^th Iteration

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.6865)(2) - 2(-0.0474)}{2 - (-0.0474)} = 1.6937$$

$f(1.6937) = (1.6937)^3 - (1.6937)^2 - 2 = -0.01$

Hence root of $x^3 - x^2 - 2 = 0$ is 1.6937.

4. **Compute the real root of $x^3 - 5x + 3 = 0$ in the interval [1, 2] by Regula Falsi method by performing four iterations. (September 2012, September 2011)**

Sol. Let $f(x) = x^3 - 5x + 3$

Since $f(1) = 1 - 5 + 3 = -1 < 0$

And $f(2) = 8 - 1 + 3 = 1 > 0$

Since $f(1) f(2) < 0$

$\therefore$ atleast one real root of $f(x)=0$ lies in the interval [1, 2]

1st iteration, Take $x_0=1,\ f(x_0)=-1$

$x_1=2,\ f(x_1)=1$

By Regula-Falsi method.

$$x_2=\frac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)}$$

$$=\frac{1\times1-2\times(-1)}{1+1}=\frac{3}{2}=1.5$$

$f(1.5)=-1.125<0$

Since $f(1.5)f(2)<0$

$\therefore$ the root of $f(x)$ lies between 1.5 and 2

2nd iteration, Take $x_0=1.5, f(x_0)=-1.125$

$x_1=2,\ f(x_1)=1$

$$x_2=\frac{(1.5)\times1-2\times(-1.125)}{1-(-1.125)}=1.764759$$

$f(1.764759)=-0.3279036<0$

Since $f(1.764759)f(2)<0$

$\therefore$ the root of $f(x)=0$ lies between 1.764759 and 2

3rd iteration, Take $x_0=1.764759,\ f(x_0)=-0.3279036$

$x_1=2,\ f(x)=1$

$$x_2=\frac{(1.764759).1-2(-0.3279036)}{1+0.3279036}$$

$$=\frac{1.764759+0.6558072}{1.3279036}=\frac{2.4205662}{1.3279036}$$

$=1.8228478$

$\therefore\ f(1.8228478)=6.056911459-9.1142239+3$

$=6.056911459-6.1142239$

$=-0.05510931<0$

Since $f(1.8228478)f(2)<0$

$\therefore$ the root of $f(x)=0$ lies between 1.8228478 and 2.

4th iteration, Take $x_0=1.8228478,\ f(x_0)=-0.05510931$

$x_1=2,\ f(x_1)=1$

$$x_2=\frac{(1.8228478)\times1-2\times(-0.05510931)}{1+0.05510931}$$

$=1.8320978$

Hence the required real root of $f(x)=0$ is 1.8320978

5. **Using Secant method and five iterations, find a root of $\cos x=xe^x$ upto four decimal places.** **(April 2012, April 2011)**

Sol. Let $f(x)=\cos x-xe^x=0$

1st approximation: Starting with $x_0=0.5$ and $x_1=1$

$f(x_0)=\cos0.5-0.5\times e^{0.5}=0.0532$

$f(x_1)=\cos1-1\times e^1=-2.1780$

$\therefore$ By secant method $x_2=\dfrac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)}$

$$=\frac{0.5\times(-2.1780)-1\times0.0532}{-2.1780-0.0532}=0.5119$$

$f(x_2)=\cos0.5119-0.5119\times e^{0.5119}=0.0177$

2nd approximation: Since we take the last two approximations.

$\therefore\ x_0=1.0,\ f(x_0)=-2.1780$

And $x_2=0.5119,\ f(x_1)=0.0177$

$$x_2=\frac{1\times0.0177-0.5119\times(-2.1780)}{0.177-(-2.1780)}=0.5158$$

$f(x_2)=\cos0.5158-0.5158e^{0.5158}=0.0058$

3rd approximation: Take $x_0=0.5119,\ f(x_0)=0.0177$

$x_1=0.5158,\ f(x_1)=0.0058$

$$x_2=\frac{0.5119\times0.0058-0.5158\times0.0177}{0.0058-0.0177}=0.5178$$

$f(x_3)=\cos0.5178-0.5178\times e^{0.5178}=0$

$\therefore$ After 3rd approximations, the function $f(x)$ becomes zero

$\therefore\ x = 0.5178$ is required root correct to four decimal places.

6. Prove that the Newton's method converges quadratically.

Sol. The general iterative formula for the Newton-Raphson method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

Let α be the actual root of $f(x)=0$, i.e., $f(\alpha)=0$ and let e_n be the error at nth stage of iteration, i.e., $e_n = x_n - \alpha$, where x_n is the approximate root at the nth iteration.

Thus, $x_n = \alpha + e_n$

Similarly, we have $x_{n+1} = \alpha + e_{n+1}$, where e_{n+1} is the error and x_{n+1} be the approximate root at $(n+1)$ th iteration.

Now, using the values of x_n and x_{n+1} in (1), we have

$$\alpha + e_{n+1} = \alpha + e_n - \frac{f(\alpha+e_n)}{f'(\alpha+e_n)} \qquad (2)$$

Expanding $f(\alpha+e_n)$ and $f'(\alpha+e_n)$ in Taylor's series around α, we get

$$e_{n+1} = e_n - \frac{\left[f(\alpha)+e_nf'(\alpha)+\frac{e_n^2}{2}f''(\alpha)+.....\right]}{\left[f'(\alpha)+e_nf''(\alpha)+\frac{e_n^2}{2}f'''(\alpha)+.....\right]}$$

Neglecting 3rd and higher order derivatives,

$$e_{n+1} = e_n - \frac{\left[f(\alpha)+e_nf'(\alpha)+\frac{e_n^2}{2}f''(\alpha)\right]}{\left[f'(\alpha)+e_nf''(\alpha)\right]}$$

Now as α is a root of $f(x)=0$

$f(x)=0$ at $x=\alpha$

$$\Rightarrow\ e_{n+1} = e_n - \left[\frac{e_nf'(\alpha)+\frac{e_n^2}{2}f''(\alpha)}{f'(\alpha)+e_nf''(\alpha)}\right]$$

$$\text{or} \ = \frac{\frac{1}{2}e_n^2f''(\alpha)}{f'(\alpha)+e_nf''(\alpha)}$$

$$\text{or} \ = \frac{1}{2}\frac{e_n^2f''(\alpha)}{f'(\alpha)\left[1+\frac{e_nf''(\alpha)}{f'(\alpha)}\right]}$$

$$\text{or} \ = \frac{1}{2}\frac{e_n^2f''(\alpha)}{f'(\alpha)}\left[1+\frac{e_nf''(\alpha)}{f'(\alpha)}\right]^{-1}$$

Expanding by Binomial theorem and neglecting term involving $e_n^2, e_n^3....$ in expansion.

$$\text{or} \ = \frac{1}{2}\frac{e_n^2f''(\alpha)}{f'(\alpha)}\left[1-\frac{e_nf''(\alpha)}{f'(\alpha)}\right]$$

$$e_{n+1} = \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + O(e_n^2)$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{2f'(\alpha)}, \text{ on ignoring } O(e_n^2)$$

$$\frac{e_{n+1}}{e_n^2} = C, \text{ where } C = \frac{f''(\alpha)}{2f'(\alpha)}.$$

Thus by definition, the order of convergence of Newton-Raphson method is 2, *i.e.*, Newton-Raphson method is of quadratic convergent.

7. Solve by false position method the equation $x^3+x-1=0$ in four steps upto three decimal places. **(September 2010)**

Sol. Let $f(x)=x^3+x-1$

Since $f(0)=-1$ and $f(1)=1$

Since $f(0)f(1)<0$

$\therefore\ f(x)=0$ has atleast one real root in the interval [0, 1]

1st iteration, Take $x_0=0$, $f(x_0)=-1$

$x_1=1$, $f(x_1)=1$

By Regula-Falsi method,

$$x_2 = \frac{x_0f(x_1)-x_1f(x_0)}{f(x_1)-f(x_0)} = \frac{0\times1-1\times(-1)}{1+1} = \frac{1}{2} = 0.5$$

$f(0.5) = -0.375 < 0$

Since $f(0.5)f(1)<0$

$\therefore$ the root of $f(x)=0$ lies between 0.5 and 1

2nd iteration, Take $x_0 = 0.5,\ f(x_0) = -0.375$

$x_1 = 1,\ f(x) = 1$

$$x_2 = \frac{(0.5)(1)-(1)(-0.375)}{1+0.375} = 0.636$$

$f(0.636) = -0.1067 < 0$

Since $f(0.636)f(1) < 0$

$\therefore$ the root of $f(x)=0$ lies between 0.636 and 1

3rd iteration, Take $x_0 = 0.636,\ f(x_0) = -0.1067$

$x_1 = 1,\ f(x_1) = 1$

$$x_2 = \frac{(0.636)(1)-(1)(-0.1067)}{1+0.1067} = 0.6711$$

$\therefore\ f(0.6711) = -0.02665 < 0$

Since $f(0.6711)f(1) < 0$

$\therefore$ the root lies between 0.6711 and 1

4th iteration, Take $x_0 = 0.6711,\ f(x_0) = -0.02665$

$x_1 = 1,\ f(x_1) = 1$

$$x_2 = \frac{(0.6711)(1)-(1)(-0.02665)}{1+0.02665} = 0.6796$$

$f(0.6796) = -0.00652 < 0$

Since $f(0.6796)f(1) < 0$

$\therefore$ root lies between 0.6796 and 1

5th iteration, Take $x_0 = 0.6796,\ f(x_0) = -0.00652$

$x_1 = 1,\ f(x_1) = 1$

$$x_2 = \frac{(0.6796)(1)-(1)(-0.00652)}{1+0.00652} = 0.68168$$

$f(0.68168) = -0.0015517$

Since $f(0.68168)f(1) < 0$

$\therefore$ root lies between 0.68168 and 1

6th iteration, Take $x_0 = 0.68168,\ f(x_0) = -0.0015517$

$x_1 = 1,\ f(x_1) = 1$

$$x_2 = \frac{(0.68168)(1)-(1)(-0.0015517)}{1+0.0015517} = 0.68217$$

$\therefore\ f(0.68217) = -0.000378 < 0$

Since $f(0.68217)f(1) < 0$

$\therefore$ the root lies between 0.68217 and 1

7th iteration, Take $x_0 = 0.68217\ f(x_0) = -0.000378$

$x_1 = 1,\ f(x_1) = 1$

$$x_2 = \frac{(0.68217)(1)-(1)(-0.000378)}{1+0.000378} = 0.6822$$

We observe that there is no change at first three decimal places in the last two iterations. Hence the root is 0.682 correct to three decimal places.

8. Use Newton-Raphson's method to find the root of the equation $x\sin x + \cos x = 0$ which is nearer $x = \pi$. (April 2010)

Sol. We have $f(x) = x\sin x + \cos x = 0$

$\therefore\ f'(x) = x\cos x + \sin x - \sin x = x\cos x$

By Newton – Raphson's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

$$= \frac{x_n^2 \cos x_n - x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

$$x_1 = \frac{x_0^2 \cos x_0 - x_0 \sin x_0 - \cos x_0}{x_0 \cos x_0} = \frac{\pi^2 \cos\pi - \pi\sin\pi - \cos\pi}{\pi\cos\pi} = 2.824$$

$$x_2 = \frac{x_1^2 \cos x_1 - x_1 \sin x_1 - \cos x_1}{x_1 \cos x_1}$$

$$= \frac{(2.824)^2(-0.95)-(2.824)(0.3123)+(0.95)}{(2.824)(-0.95)} = 2.8022$$

$$x_3 = \frac{x_2^2 \cos x_2 - x_2 \sin x_2 - \cos x_2}{x_2 \cos x_2}$$

$$= \frac{(2.8022)^2(-0.9429)-(2.8022)(0.3329)+(0.9429)}{(2.8022)(-0.9429)} = 2.797$$

$$x_4 = \frac{x_3^2 \cos x_3 - x_3 \sin x_3 - \cos x_3}{x_3 \cos x_3}$$

$$= \frac{(2.797)^2(-0.9412) - (2.797)(0.3378) + (0.9412)}{(2.797)(-0.9412)} = 2.7983$$

$$x_5 = \frac{x_4^2 \cos x_4 - x_4 \sin x_4 - \cos x_4}{x_4 \cos x_4}$$

$$= \frac{(2.7983)^2(-0.9416) - (2.7983)(0.3365) + (0.9416)}{(2.7983)(-0.9416)} = 2.7984$$

Hence root of $x \sin x + \cos x = 0$ is 2.7984

9. Develop a recurrence formula for finding the value of $\sqrt{n}$, using Newton Raphson method and hence compute $\sqrt{32}$ upto four decimal places. (September 2009)

Sol. Let $x = \sqrt{n} \Rightarrow x^2 - n = 0$

$f(x) = x^2 - n \quad \therefore f'(x) = 2x$

By Newton–Raphson Method

$$x_{n+1} = x_n - \frac{(x_n^2 - n)}{2x_n} = \frac{x_n^2 + n}{2x_n} = \frac{1}{2}\left(x_n + \frac{n}{x_n}\right)$$

Taking initial approximation $x_0 = 5.5$

$$x_1 = \frac{1}{2}\left(5.5 + \frac{32}{5.5}\right) = 5.6590$$

$$x_2 = \frac{1}{2}\left(5.659 + \frac{32}{5.659}\right) = 5.6568$$

$$x_3 = \frac{1}{2}\left(5.6568 + \frac{32}{5.6568}\right) = 5.6568$$

Hence $\sqrt{32} = 5.6568$

10. Using Newton-Raphson formula, find the iterative formula for finding the cube root of a positive integer n. Use it to find the cube root of 128 upto 3 places of decimal. (April 2009)

Sol. Let $x = n^{\frac{1}{3}}$, $n > 0$ is an integer

$\therefore x^3 = n \quad \Rightarrow \quad f(x) = x^3 - n = 0$

Now Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$\therefore$ Iterative formula is

$$x_{n+1} = x_n - \frac{(x_n^3 - n)}{3x_n^2} = \frac{2x_n^3 + n}{3x_n^2}$$

Cube root of 128:

Let $x_0 = 5$, Here $n = 128$

$$\Rightarrow x_1 = \frac{2x_0^3 + 128}{3x_0^2}$$

$$= \frac{2(5)^3 + 128}{3(5)^2} = \frac{378}{75} = 5.04$$

$$x_2 = \frac{2x_1^3 + 128}{3x_1^2}$$

$$= \frac{2(5.04)^3 + 128}{3(5.04)^2} = 5.0396$$

$$x_3 = \frac{2x_2^3 + 128}{3x_2^2}$$

$$= \frac{2(5.0396)^3 + 128}{3(5.0396)^2} = 5.0396$$

Hence $\sqrt[3]{128} = 5.0396$

2

INTERPOLATION

1. Using Hermite's interpolation formula, evaluate log(27) from the following data: **(September 2013)**

$x:$	2.5	3.0
$y=\log x$	0.91629	1.09861
$\frac{dy}{dx}=\frac{1}{x}$	0.4000	0.3333

Sol. Here $x_0=2.5,\quad x_1=3.0$

$y_0=0.91629,\quad y_1=1.09861$

$y_0'=0.4000,\quad y_1'=0.3333$

$n=1$

The hermite interpolating polynomial is

$$P(x)=\sum_{i=0}^{1}\left[1-2(x-x_i)l_i'(x_i)\right]\left[l_i(x)\right]^2 y_i+\sum_{i=0}^{1}(x-x_i)\left[l_i(x)\right]^2 y_i'$$

$$=\left[1-2(x-x_0)l_0'(x_0)\right]\left[l_0(x)\right]^2 y_0+\left[1-2(x-x_1)l_1'(x_1)\right]\left[l_1(x)\right]^2 y_1$$
$$+(x-x_0)\left[l_0(x)\right]^2 y_0'+(x-x_1)\left[l_1(x)\right]^2 y_1' \qquad (1)$$

Now $l_0(x)=\frac{x-x_1}{x_0-x_1}=\frac{x-3}{2.5-3}=6-2x$

$$l_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x-2.5}{3-2.5}=2x-5$$

$l_0'(x)=-2 \qquad \Rightarrow l_0'(x_0)=-2$

$l_1'(x)=2 \qquad \Rightarrow l_1'(x_1)=2$

Putting these values in (1), we get

$$P(x)=\left[1-2(x-2.5)(-2)\right]\left[6-2x\right]^2(0.91629)+\left[1-2(x-3)(2)\right]$$

$$\left[2x-5\right]^2(1.09861)+(x-2.5)(6-2x)^2(0.4)$$
$$+(x-3)(2x-5)^2(0.3333)$$

Putting $x=2.7$

$$P(2.7)=[1.8][0.6]^2(0.91629)+[2.2][0.4]^2(1.09861)$$
$$+(0.2)(0.6)^2(0.4)+(-0.3)(0.4)^2(0.3333)$$
$$=0.59375+0.38671+0.02880-0.01599$$
$$=0.99327$$

$\Rightarrow \log_e 2.7=0.99327(approx)$

Also $\log_e 2.7=\log_e(2.7)(10)$

$=\log_e 10+\log_e(2.7)$

$=(2.303)+(.99327)$

$=3.29627(approx)$

2. Show that the nth divided difference can be expressed as the quotient of two determinants each of order $(n+1)$. **(September 2013)**

Sol. Consider the first divided difference of the function $f(x)$.

$$[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$=\begin{vmatrix} f(x_1) & f(x_0) \\ 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & x_0 \\ 1 & 1 \end{vmatrix}$$

Again consider the second divided difference

$$[x_0,x_1,x_2]=\frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$$

$$=\frac{f(x_2)-f(x_1)}{(x_2-x_1)(x_2-x_0)}-\frac{f(x_1)-f(x_0)}{(x_1-x_0)(x_2-x_0)}$$

$$=\frac{f(x_0)}{(x_1-x_0)(x_2-x_0)}-\frac{1}{(x_2-x_0)}\left[\frac{1}{x_2-x_1}+\frac{1}{x_1-x_0}\right]f(x_1)+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$=\frac{f(x_0)}{(x_1-x_0)(x_2-x_0)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{(x_2 - x_1) f(x_0) + (x_0 - x_2) f(x_1) + (x_1 - x_0) f(x_2)}{(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)}$$

$$= \frac{(x_1 - x_2) f(x_0) + (x_2 - x_0) f(x_1) + (x_0 - x_1) f(x_2)}{-(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)}$$

$$= \begin{vmatrix} f(x_0) & f(x_1) & f(x_2) \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\left(\because \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix} = -(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) \right)$$

Similarly the third divided difference

$$[x_0, x_1, x_2, x_3] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$+ \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_1)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

$$= \sum \left[f(x_0) \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \right]$$

$$= \begin{vmatrix} f(x_0) & f(x_1) & f(x_2) & f(x_3) \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \div \begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

Continuing in this way, the *n*th order divided difference can expressed as the quotient of two determinants each of order $(n+1)$.

3. **Discuss the error in Lagrange's interpolation formula.** **(April 2013, 2011)**

Sol. Let the function y(x), defined by the (n + 1) points (x_i, y_i), i = 0, 1, 2, ……, n be continuous and differentiable (n + 1) times and let y(x) be approximated by a polynomial $P_n(x)$ of degree at most n such that

$P_n(x_i) = y_i$, i = 0, 1, 2, ……, n (1)

$\Rightarrow y(x) - P_n(x) = 0$ for $x = x_0, x_1, \ldots, x_n$ (2)

So, let $y(x) - P_n(x) = A(x - x_0)(x - x_1) \ldots (x - x_n)$ (3)

Where A is to be determined such that (2) holds for any $x = x'$, $x_0 < x' < x_n \ldots$; $x' \neq x_0, x_1, \ldots, x_n$

Putting $x = x'$ in (3), we have

$$y(x') - P_n(x') = A(x' - x_0)(x' - x_1) \ldots (x' - x_n)$$

$$\therefore A = \frac{y(x') - P_n(x')}{(x' - x_0)(x' - x_1) \ldots (x' - x_n)} \qquad (4)$$

Define a function

$$F(x) = y(x) - P_n(x) - A(x - x_0)(x - x_1) \ldots (x - x_n) \qquad (5)$$

where A is given by (4),

Since $F(x_0) = F(x_1) = \ldots = F(x_n) = F(x') = 0$

$\therefore$ F(x) vanishes (n + 2) times in the interval $x_0 \le x \le x_n$.

Also F(x) is continuous and differentiable (n + 1) times

$\therefore$ By repeated application of Rolle's theorem

$F'(x)$ must vanish (n + 1) times, $F''(x)$ must vanish n times etc. in the interval $x_0 \le x \le x_n$.

In particular $F^{(n+1)}(x)$ must vanish once in the interval $x_0 < x < x_n$.

Let $F^{(n+1)}(\xi) = 0$, $x_0 < \xi < x_n$ (6)

Differentiating (5) (n + 1) times w.r.t. x

$$F^{(n+1)}(x) = y^{(n+1)}(x) - 0 - A(n+1)!$$

[$\because$ $P_n(x)$ and $(x - x_0)(x - x_1) \ldots (x - x_n)$ are polynomial of degree n and (n + 1) respectively]

$$\Rightarrow F^{(n+1)}(\xi) = y^{(n+1)}(\xi) - A(n+1)!$$

$$\Rightarrow 0 = y^{(n+1)}(\xi) - A(n+1)! \qquad \text{[By (6)]}$$

$$\therefore A = \frac{y^{(n+1)}(\xi)}{(n+1)!} \qquad (7)$$

From (4) and (7), equating two values of A

$$\frac{y(x') - P_n(x')}{(x' - x_0)(x' - x_1) \ldots (x' - x_n)} = \frac{y^{(n+1)}(\xi)}{(n+1)!}$$

Or $y(x') - P_n(x') = \dfrac{y^{(n+1)}(\xi)}{(n+1)!}(x' - x_0)(x' - x_1) \ldots (x' - x_n)$

Changing x' to x, we get

$$y(x)-P_n(x)=\frac{(x-x_0)(x-x_1)...(x-x_n)}{(n+1)!}y^{(n+1)}(\xi),\ x_0<x<x_n$$

$$\therefore\ Error=\frac{(x-x_0)(x-x_1)...(x-x_n)}{(n+1)!}y^{(n+1)}(\xi),\ x_0<x<x_n$$

4. Using Newton's divided difference formula, prove that: **(April 2013)**

$$f(x)=f(0)+x\Delta f(-1)+\frac{(x+1)x}{2!}\Delta^2 f(-1)+\frac{(x+1)x(x-1)}{3!}\Delta^3 f(-3)+.......$$

Sol. For the arguments 0, –1, 1, –2, 2, ..., the divided difference formula is

$$f(x)=y_0+(x-x_0)\Delta_d y_0+(x-x_0)(x-x_1)\Delta^2_d y_0+(x-x_0)(x-x_1)(x-x_2)\Delta^3_d y_0+......$$

i.e. $f(x)=f(0)+(x-0)\Delta_d y_0+(x-0)(x-(-1))\Delta^2_d y_0$

$$+(x-0)(x-(-1))(x-1)\Delta^3_d y_0+......$$

i.e. $f(x)=f(0)+x\Delta_d y_0+x(x+1)\Delta^2_d y_0+x(x+1)(x-1)\ \Delta^3_d y_0+......$ (i)

Now $\Delta_d y_0=\frac{y_1-y_0}{x_1-x_0}=\frac{f(-1)-f(0)}{-1-0}=\frac{f(0)-f(-1)}{0-(-1)}=\Delta f(-1)$

$$\Delta^2_d y_0=\frac{\Delta_d y_1-\Delta_d y_0}{x_2-x_0}=\frac{1}{x_2-x_0}\left[\frac{y_2-y_1}{x_2-x_1}-\frac{y_1-y_0}{x_1-x_0}\right]$$

$$=\frac{1}{1-0}\left[\frac{f(1)-f(-1)}{1-(-1)}-\frac{f(-1)-f(0)}{-1-0}\right]=\frac{f(1)-f(0)+2f(-1)-2f(0)}{2}$$

$$=\frac{f(1)-2f(0)+f(-1)}{2}$$

or $\Delta^2_d y_0=\frac{E^2f(-1)-2Ef(-1)+f(-1)}{2}=\frac{(E^2-2E+1)f(-1)}{2}=\frac{(E-1)^2f(-1)}{2}$

$$=\frac{\Delta^2 f(-1)}{2!}\qquad[\because\Delta\equiv E-1]$$

$$\Delta^3_d y_0=\frac{\Delta^2_d y_1-\Delta^2_d y_0}{x_3-x_0}=\frac{1}{x_3-x_0}\left[\frac{\Delta_d y_2-\Delta_d y_1}{x_3-x_1}-\frac{f(1)-2f(0)+f(-1)}{2}\right]$$

(Using (ii))

$$=\frac{1}{-2-0}\left[\frac{1}{-2-(-1)}\left(\frac{y_3-y_2}{x_3-x_2}-\frac{y_2-y_1}{x_2-x_1}\right)-\frac{f(1)-2f(0)+f(-1)}{2}\right]$$

$$=\frac{1}{-2}\left[-\left(\frac{f(-2)-f(1)}{-2-1}-\frac{f(1)-f(-1)}{1-(-1)}\right)-\frac{f(1)-2f(0)+f(-1)}{2}\right]$$

$$=-\frac{1}{2}\left[\frac{f(-2)-f(1)}{3}+\frac{f(-1)-f(1)}{2}-\frac{f(1)-2f(0)+f(-1)}{2}\right]$$

$$=-\frac{1}{2}\left[\frac{2f(-2)-2f(1)+3f(1)-3f(-1)-3f(1)+6f(0)-3f(-1)}{6}\right]$$

$$=\frac{2f(-2)-2f(1)-6f(-1)+6f(0)}{-12}=\frac{f(1)-f(-2)-3f(0)+3f(-1)}{6}$$

$$=\frac{E^3f(-2)-f(-2)-3E^2f(-2)+3Ef(-2)}{6}$$

$$=\frac{(E^3-1-3E^2+3E)f(-2)}{6}$$

$$=\frac{(E-1)^3f(-2)}{3!}=\frac{\Delta^3 f(-2)}{3!}\qquad[\because\Delta\equiv E-1]$$

and so on.

Substituting these value in (i), we get,

$$f(x)=f(0)+x\Delta f(-1)+\frac{(x+1)x}{2!}\Delta^2 f(-1)+\frac{(x+1)x(x-1)}{3!}\Delta^2 f(-2)+....$$

5. Given that $f(1)=4, f(2)=5, f(7)=5, f(8)=4$. Find the value of $f(6)$ by using Lagrange's formula and also find x for which $f(x)$ is max. or minimum. **(September 2012)**

Sol. Using Lagrange's formula, we get

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$=\frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)}\times 4+\frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)}\times 5$$

$$+\frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)}\times 5+\frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}\times 4$$

$$=\frac{-2}{21}(x-2)(x-7)(x-8)+\frac{1}{6}(x-1)(x-7)(x-8)$$

$$-\frac{1}{6}(x-1)(x-2)(x-8)+\frac{2}{21}(x-1)(x-2)(x-7)$$

$$=\frac{2}{21}(x-2)(x-7)[-x+8+x-1]+\frac{1}{6}(x-1)(x-8)[x-7-x+2]$$

$$=\frac{2}{3}(x-2)(x-7)-\frac{5}{6}(x-1)(x-8)$$

$$=\frac{2}{3}(x^2-9x+14)-\frac{5}{6}(x^2-9x+8)$$

$$=\frac{1}{6}(4x^2-36x+56-5x^2+45x-40)$$

$$=\frac{1}{6}(-x^2+9x+16)$$

$$\therefore\ f(x)=\frac{1}{6}(-x^2+9x+16)$$

Putting $x=6$, we get

$$f(6)=\frac{1}{6}(-36+54+16)=\frac{34}{6}=5.66$$

$$f'(x)=\frac{1}{6}(-2x+9)$$

For maximum or minimum, $f'(x)=0$

$$\therefore\ \frac{1}{6}(-2x+9)=0$$

$$\therefore\ x=\frac{9}{2}=4.5$$

6. **Find the Cubic Lagrange's interpolating polynomial from the following data:**

x :	0	1	2	5	
$f(x)$:	2	3	12	147	**(April 2012)**

Sol. Using Lagrange's formula

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(x)=\frac{(x-1)(x-2)(x-5)}{(-1)(-2)(-5)}(2)+\frac{(x)(x-2)(x-5)}{(1)(-1)(-4)}(3)$$

$$+\frac{(x)(x-1)(x-5)}{(2)(1)(-3)}(12)+\frac{(x)(x-1)(x-2)}{(5)(4)(3)}(147)$$

$$=\frac{-1}{5}(x-1)(x-2)(x-5)+\frac{3}{4}(x)(x-2)(x-5)$$

$$-2(x)(x-1)(x-5)+\frac{49}{20}(x)(x-1)(x-2)$$

$$=\frac{-1}{5}(x^3-8x^2+17x-10)+\frac{3}{4}(x^3-7x^2+10x)$$

$$-2(x^3-6x^2+5x)+\frac{49}{20}(x^3-3x^2+2x)$$

$$=x^3+x^2-x+2$$

7. **Given $\log 654=2.8156$, $\log 658=2.8182$, $\log 659=2.8189$ and $\log 661=2.8202$, find the value of log 656.** **(September 2011)**

Sol. Here we have $x_0=654,\ x_1=658,\ x_2=659,\ x_3=661$

$f(x_0)=2.8156, f(x_1)=2.8182, f(x_2)=2.8189, f(x_3)=2.8202$

where $f(x)=\log_{10}x$

Using Lagrange's formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

Putting $x=656$, we get

$$f(656)=\frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}\times(2.8156)$$

$$+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}\times(2.8182)$$

$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}\times(2.8189)$$

$$+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}\times(2.8202)$$

$$=\frac{(-2)(-3)(-5)}{(-4)(-5)(-7)}\times(2.8156)+\frac{(2)(-3)(-5)}{(4)(-1)(-3)}\times(2.8182)$$

$$+\frac{(2)(-2)(-5)}{(5)(1)(-2)}\times(2.8189)+\frac{(2)(-2)(-3)}{(7)(3)(2)}\times(2.8202)$$

$= 2.8168$ (nearly)

Hence $\log_{10} 656 = 2.8168$

8. The values of $\sin x$ are given below for different values of x. Find the value of $\sin 32°$. (September 2010, September 2009)

x	**30°**	**35°**	**40°**	**45°**	**50°**
y = sin x	**0.5000**	**0.5736**	**0.6428**	**0.7071**	**0.7660**

Sol.

x	$y=\sin x$	Δy	$X^2 y$	$\Delta^3 y$	$\Delta^4 y$
30°	0.5000				
		0.0736			
35°	0.5736		–0.0044		
		0.0692		–0.0005	
40°	0.6428		–0.0049		0
		0.0643		–0.0005	
45°	0.7071		–0.0054		
		0.0589			
50°	0.7660				

Here $x_0 = 30°$, $y_0 = 0.5000$, $h = 5°$ *and* $x = 32°$

$$\therefore\ p = \frac{x-x_0}{h} = \frac{32°-30°}{5°} = 0.4$$

By Newton's forward interpolation formula, we have

$$f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

$$\therefore\ f(32°) = 0.5000 + (0.4)(0.0736) + \frac{(0.4)(0.4-1)}{2!}$$

$$\times(-0.0044) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}\times(-0.0005)$$

$= 0.5000 + 0.2944 + 0.000528 - 0.00032$

$= 0.299$

9. Find the polynomial of the lowest possible degree which assumes the values 1245, 33, 5, 9 and 1355 at x = –4, –1, 0, 2 and 5. Also find the value of the polynomial at abscissa 1. (September 2010)

Sol. Since the magnitudes of x are not equi-distant, therefore, we use Newton's divided difference formula

The divided difference table is

	$y=f(x)$	$\Delta_d f(x)$	$\Delta_d^2 f(x)$	$\Delta_d^3 f(x)$	$\Delta_d^4 f(x)$
	1245				
		$\frac{33-1245}{-1-(-4)}=-404$			
	33		$\frac{-28-(-404)}{0-(-4)}=94$		
		$\frac{5-33}{0-(-1)}=-28$		$\frac{10-94}{2-(-4)}=-14$	
	5		$\frac{2-(-28)}{2-(-1)}=10$		$\frac{13-(-14)}{5-(-4)}=3$
		$\frac{9-5}{2-0}=2$		$\frac{88-10}{5-(-1)}=13$	
	9		$\frac{442-2}{5-0}=88$		
		$\frac{1335-9}{5-2}=442$			
	1335				

$\therefore$ By Newton's divided difference formula

$$f(x)=f(x_0)+(x-x_0)\Delta_d f(x_0)+(x-x_0)(x-x_1)\Delta_d^2 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)\Delta_d^3 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta_d^4 f(x_0)$$
$$=1245+(x+4)(-404)+(x+4)(x+1)94$$
$$+(x+4)(x+1)(x-0)(-14)+(x+4)(x+1)(x-0)(x-2)(3)$$
$$=3x^4-5x^3+6x^2-14x+5$$

10. Using Lagrange's Interpolation formula express the rational function $f(x)=\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as the sum of partial fractions. (April 2010)

Sol. Consider the numerator x^2+6x-1

Let $f(x)=x^2+6x-1$

Tabulating values of $f(x)$ for $x=-1,1,4,6$ we get

x	:	–1	1	4	6
f (x)	:	–6	6	39	71

Using Lagrange's formula

$$f(x)=\frac{(x-1)(x-4)(x-6)}{(-2)(-5)(-7)}\times(-6)+\frac{(x+1)(x-4)(x-6)}{(2)(-3)(-5)}\times(6)$$
$$+\frac{(x+1)(x-1)(x-6)}{(5)(3)(-2)}\times(39)+\frac{(x+1)(x-1)(x-4)}{(7)(5)(2)}\times(71)$$
$$=\frac{3}{35}(x-1)(x-4)(x-6)+\frac{1}{5}(x+1)(x-4)(x-6)$$
$$-\frac{13}{10}(x+1)(x-1)(x-6)+\frac{71}{70}(x+1)(x-1)(x-4)$$
$$\therefore \frac{(x^2+6x-1)}{(x^2-1)(x-4)(x-6)}=\frac{3}{35(x+1)}+\frac{1}{5(x-1)}-\frac{13}{10(x-4)}+\frac{71}{70(x-6)}$$

11. Find the divided differences of various orders for the following data:

x	:	–3	–1	0	3	5
f(x)	:	–30	–22	–12	330	3458

(April 2010)

Sol.

x	$f(x)$	$\Delta_d f(x)$	$\Delta_d^2 f(x)$	$\Delta_d^3 f(x)$	$\Delta_d^4 f(x)$
-3	–30				
		$\frac{-22+30}{-1+3}=4$			
-1	–22		$\frac{10-4}{0+3}=2$		
		$\frac{-12+22}{0+1}=10$		$\frac{26-2}{3+3}=4$	
0	–12		$\frac{114-10}{3+1}=26$		$\frac{44-4}{5+3}=8$
		$\frac{330+12}{3-0}=114$		$\frac{290-26}{5+1}=44$	
3	330		$\frac{1564-114}{5-0}=290$		
		$\frac{3458-330}{5-3}=1564$			
5	3458				

12. Given the following data, obtain the Hermite interpolating polynomial

x	$f(x)$	$f'(x)$
0	0	2
6	–6	–4

(September 2009)

Sol. Using Hermite interpolating polynomial

$$P(x)=\sum_{i=0}^{1}\left[1-2(x-x_i)l_i'(x_i)\right]\left[l_i(x)\right]^2 f(x_i)+\sum_{i=0}^{1}(x-x_i)\left[l_i(x)\right]^2 f'(x_i)$$
$$=\left[1-2(x-x_0)l_0'(x_0)\right]\left[l_0(x)\right]^2 f(x_0)+\left[1-2(x-x_1)l_1'(x_1)\right]\left[l_1(x)\right]^2 f(x_1)$$
$$+(x-x_0)\left[l_0(x)\right]^2 f'(x_0)+(x-x_1)\left[l_1(x)\right]^2 f'(x_1)$$
$$l_0(x)=\frac{x-x_1}{x_0-x_1}=\frac{x-6}{0-6}=\frac{1}{6}(6-x)\Rightarrow l_0'(x)=\frac{-1}{6}\Rightarrow l_0'(x_0)=\frac{-1}{6}$$

$$l_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x-0}{6-0}=\frac{x}{6}\Rightarrow l_1'(x)=\frac{1}{6}\Rightarrow l_1'(x_1)=\frac{1}{6}$$

$$P(x)=\left[1-2(x-0)\left(\frac{-1}{6}\right)\right]\left(\frac{(6-x)}{6}\right)^2(0)$$

$$+\left[1-2(x-6)\left(\frac{1}{6}\right)\right]\left(\frac{x}{6}\right)^2(-6)$$

$$+(x)\left(\frac{6-x}{6}\right)^2(2)+(x-6)\left(\frac{x}{6}\right)^2(-4)$$

$$=0+\left(\frac{9-x}{3}\right)\left(\frac{x^2}{6}\right)(-1)+\frac{x(6-x)^2}{18}-\frac{x^2(x-6)}{9}$$

$$=\frac{x^3-9x^2+x^3+36x-12x^2-2x^3+12x^2}{18}$$

$$=-\frac{x^2}{2}+2$$

13. Use appropriate interpolation method to estimate the number of students who obtained marks between 40 and 45, from the following table: (April 2009)

Marks:	**30–40**	**40–50**	**50–60**	**60–70**	**70–80**
No. of Students	**31**	**42**	**51**	**35**	**31**

Sol. The difference table is

Marks (x)	No. of Students (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	31				
		42			
Below 50	31 + 42 = 73		9		
		51		–25	
Below 60	73 + 51 = 124		–16		37
		35		12	
Below 70	124 + 35 = 159		–4		
		31			
Below 80	159 + 31 = 190				

Here $x_0=40$, $h=10$ and $x=45$

$$\therefore\ p=\frac{x-x_0}{h}=\frac{45-40}{10}=\frac{5}{10}=0.5$$

By Newton's forward interpolation formula

$$f(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0+\ldots\ldots$$

$$\therefore\ f(45)=31+(0.5)(42)+\frac{(0.5)(0.5-1)}{2}\times(9)+\frac{(0.5)(0.5-1)(0.5-2)}{6}\times(-25)+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}\times(37)$$

$=31+42-125-1.5625-1.4453$

$=47.8672$

$=48$ nearly

$\therefore$ Number of students getting marks between 40 and 45 = 48 – 31 = 17

14. Find the polynomial of least degree which attains the prescribed value at the given points:

x	**:**	**2**	**3**	**4**	**5**
f (x)	**:**	**7**	**9**	**12**	**16**

Sol.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	7			
		2		
3	9		1	
		3		0
4	12		1	
		4		
5	16			

Using Newton forward Interpolation formula, interpolating polynomial is given by

$$f(x)=f_0+p\Delta f_0+\frac{p(p-1)}{2!}\Delta^2 f_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 f_0$$

where $p = \dfrac{x - x_0}{h}$ and $f_0 = y_0$

Here, $p = \dfrac{x-2}{1} = x - 2$

$$\therefore f(x) = 7 + (x-2)(2) + \frac{(x-2)(x-3)}{2}(1) + \frac{(x-2)(x-3)(x-4)}{6}(0)$$

$$= 7 + 2x - 4 + \frac{(x^2 - 5x + 6)}{2}$$

$$= \frac{6 + 4x + x^2 - 5x + 6}{2}$$

$$= \frac{x^2 - x + 12}{2}$$

3

NUMERICAL DIFFERENTIATION

1. **Using the following data, find x for which y is minimum:** **(April 2013)**

x =	**0.60**	**0.65**	**0.70**	**0.75**
y =	**0.6221**	**0.6155**	**0.6138**	**0.6170**

Also find the minimum value of y.

Sol.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
.60	.6221			
		-0.0066		
.65	.6155		.0049	
		-0.0017		0
.70	.6138		.0049	
		0.0032		
.75	.6170			

$h = 0.05$, $x_0 = 0.60$

By Newton's forward formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots..$$

where $x = x_0 + ph$

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \ldots..$$

for minimum value of $y, \dfrac{dy}{dp} = 0$

$\Rightarrow -0.0066+\frac{2p-1}{2}(.0049)=0$

$\Rightarrow (2p-1)(49)=132$

$\Rightarrow p=\frac{181}{98}=1.8469$

$\Rightarrow x=0.60+(1.8469)(0.05)=0.692$

Using Newton forward Interpolation formula

$y=.6221+(1.8469)(-0.0066)+\frac{(1.8469)(.8469)}{2}(.0049)$

$=.6221-0.0121895+0.0038321$

$=.6137$

$\therefore$ Minimum value of y occurs at $x=0.692$ and minimum value $y=0.6137$

2. From the following data, find $\frac{dy}{dx}$ at $x=1.05$: (September 2012)

x = 1.00	1.05	1.10	1.15	1.20	1.25	1.30
y = 1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

Sol. As derivatives are required at $x=1.05$ which is near the beginning of the table, so we use Newton's forward formula.

Here we take $x_0=1.05$, $h=0.05$

The different table is

x	$10^5 y$	$10^5\Delta y$	$10^5\Delta^2 y$	$10^5\Delta^3 y$	$10^5\Delta^4 y$	$10^5\Delta^5 y$
1.05	102470					
		2411				
1.10	104881		−54			
		2357		3		
1.15	107238		−51		1	
		2306		4		−3
1.20	109544		−47		−2	
		2259		2		
1.25	111803		−45			
		2214				
1.30	114017					

By Newton's forward formula

$$\left[\frac{dy}{dx}\right]_{x=x_0}=\frac{1}{h}\left[\Delta y_0-\frac{\Delta^2 y_0}{2}+\frac{\Delta^3 y_0}{3}-\frac{\Delta^4 y_0}{4}+\frac{\Delta^5 y_0}{5}\text{..........}\right]$$

$$\therefore 10^5\left[\frac{dy}{dx}\right]_{x=1.05}=\frac{1}{0.05}\left[2411+\frac{54}{2}+\frac{3}{3}-\frac{1}{4}+\frac{3}{5}\right]$$

$=\frac{1}{0.05}[2411+27+1-0.25+0.6]$

$=\frac{24439.35}{0.05}$

$=48787$

$\therefore \left[\frac{dy}{dx}\right]_{x=1.05}=0.48787$

3. Find $f'(2.5)$ from the following table: (April 2012)

x	:	1.5	1.9	2.5	3.2	4.3	5.9
f(x)	:	3.375	6.059	13.625	29.368	73.907	196.579

Sol. Since values of x are not equally spaced, we shall use Newton's divided difference formula.

x	$f(x)$	$\Delta_d f(x)$	$\Delta_d^2 f(x)$	$\Delta_d^3 f(x)$	$\Delta_d^4 f(x)$	$\Delta_d^5 f(x)$
1.5	3.375					
		6.71				
1.9	6.059		4.882			
		12.61		19.382		
2.5	13.625		6.29		6.187	
		22.49		8.591		2.589
3.2	29.368		15.997		7.179	
		40.49		14.51		
4.3	73.907		28.938			
		76.67				
5.9	196.579					

$f(x)=f(x_0)+(x-x_0)\Delta_d f(x_0)+(x-x_0)(x-x_1)\Delta_d^2 f(x_0)$

$+(x-x_0)(x-x_1)(x-x_2)\Delta_d^3 f(x_0)$

$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta_d^4 f(x_0)$

$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\Delta_d^5 f(x_0)$$

$$f'(x)=\Delta_d f(x_0)+[(x-x_0)+(x-x_1)]\Delta_d^2 f(x_0)+[(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_2)+(x-x_0)(x-x_1)]\Delta_d^3 f(x_0)+[(x-x_1)(x-x_2)(x-x_3)$$
$$+(x-x_0)(x-x_2)(x-x_3)+(x-x_0)(x-x_1)(x-x_3)$$
$$+(x-x_0)(x-x_1)(x-x_2)]\Delta_d^4 f(x_0)$$
$$[(x-x_1)(x-x_2)(x-x_3)(x-x_4)+(x-x_0)(x-x_2)(x-x_3)(x-x_4)$$
$$+(x-x_0)(x-x_1)(x-x_3)(x-x_4)+(x-x_0)(x-x_1)(x-x_2)(x-x_4)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)]\Delta_d^5 f(x_0)$$

Hence

$$f'(2.5)=6.71+[1+0.6](4.822)+[(1)(.6)](19.382)$$
$$+[(1)(.6)(-0.7)](6.187)+[(1)(.6)(-0.7)(-1.8)](2.589) \qquad \because x-x_2=0$$
$$=6.71+(1.6)(4.882)+(.6)(19.382)+(-.42)(6.187)+(.756)(2.589)$$
$$=25.509144$$

4. **Given the following pair of values of x and y:** **(September 2011)**

x:	1	2	4	8	10
$y=f(x)$:	0	1	5	21	27

Determine numerically the first derivative at x = 4.

Sol. Since the values of x are not equally spaced, we shall use Newton's divided difference formula.

Divide difference table is

x	$y=f(x)$	$\Delta_d f(x)$	$\Delta_d^2 f(x)$	$\Delta_d^3 f(x)$	$\Delta_d^4 f(x)$
1	0				
		1			
2	1		0.3333		
		2		0	
4	5		0.3333		–0.0069
		4		–0.0625	
8	21		–0.1667		
		3			
10	27				

Since fourth divided difference is constant, the Newton's divided difference formula is

$$f(x)=f(x_0)+(x-x_0)\Delta_d f(x_0)+(x-x_0)(x-x_1)\Delta_d^2 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)\Delta_d^3 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta_d^4 f(x_0) \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$f'(x)=\Delta_d f(x_0)+[(x-x_0)+(x-x_1)]\Delta_d^2 f(x_0)$$
$$+[(x-x_0)(x-x_1)+(x-x_0)(x-x_2)+(x-x_1)(x-x_2)]\Delta_d^3 f(x_0)$$
$$+[(x-x_0)(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_3)$$
$$+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]\Delta_d^4 f(x_0) \qquad (2)$$

Putting $x_0=1$, $x_1=2$, $x_2=4$, $x_3=8$, $x=4$ and the $\Delta_d f(x_0), \Delta_d^2 f(x_0), \Delta_d^3 f(x_0), \Delta_d^4 f(x_0)$ in (2), we get

$$f'(4)=1+(3+2)(0.3333)+[(4-1)(4-2)+(4-1)(4-4)$$
$$+(4-2)(4-4)](0)+[(4-1)(4-2)(4-4)$$
$$+(4-1)(4-2)(4-8)+(4-1)(4-4)(4-8)$$
$$+(4-2)(4-4)(4-8)](-0.0069)$$
$$=1+1.6665+0+(0-24+0+0)(-0.0069)$$
$$=1+1.6665+0.1656$$
$$=2.8321$$

5. **Find the first and second derivatives of $y=f(x)$ at $x=0$, given that** **(September 2010)**

x	0	1	2	3	4	5
y = f(x)	4	8	15	7	6	2

Sol. **The difference table is**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		4				
1	8		3			
		7		–18		

2	15		−15		40	
		−8		22		−72
3	7		7		−32	
		−1		−10		
4	6		−3			
		−4				
5	2					

From Newton's forward formula, we have

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} \ldots\ldots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{1}\left[4 - \frac{3}{2} - \frac{18}{3} - \frac{40}{4} - \frac{72}{5}\right]$$

$$= 4 - 1.5 - 6 - 10 - 14.4 = -27.9$$

and $$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\ldots\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{(1)^2}\left[3 - (-18) + \frac{11}{12}(40) - \frac{5}{6}(-72)\right]$$

$$= 3 + 18 + 36.6667 + 60 = 117.667$$

6. **Using the following table compute $f'(16)$ and $f''(16)$:** **(April 2010)**

x	**15**	**17**	**19**	**21**	**23**	**25**
f(x)	**3.873**	**4.123**	**4.359**	**4.583**	**4.796**	**5.000**

Sol.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	3.873					
		0.250				
17	4.123		−0.014			
		0.236		0.002		
19	4.359		−0.012		−0.001	
		0.224		0.001		0.002
21	4.583		−0.011		0.001	
		0.213		0.002		
23	4.796		−0.009			
		0.204				
25	5.000					

To Calculate $\left[\frac{dy}{dx}\right]_{x=16}$ and $\left[\frac{d^2y}{dx^2}\right]_{x=16}$ $\quad \therefore p = \frac{16-15}{2} = \frac{1}{2}$

From Newton's forward formula

$$\left[\frac{dy}{dx}\right]_{x=16} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0\right.$$

$$+ \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0$$

$$\left. + \frac{(5p^4-40p^3+105p^2-100p+24)}{5!}\Delta^5 y_0\right]$$

$$= \frac{1}{2}\left[(0.250) + \frac{(0)}{2!}(-0.014) + \frac{(-0.25)}{3!}(0.002) + \frac{(1)}{4!}(-0.001) + \frac{(-4.4375)}{5!}(0.002)\right]$$

$$= 0.1249$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=16} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{(12p^2-36p+22)}{4!}\Delta^4 y_0\right.$$

$$\left. + \frac{(20p^3-120p^2+210p-100)}{5!}\Delta^5 y_0\right]$$

$$= \frac{1}{4}\left[(-0.014) + (-0.5)(0.02) + \frac{7}{4!}(-0.001) + \frac{(-22.5)}{5!}(0.002)\right]$$

$$= \frac{1}{4}[-0.01566] = -0.003915$$

7. From the following table, find the value of x for which y is minimum and find this value of y. **(September 2009)**

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9638	0.9855	0.9975	0.9996

Sol.

	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x	.9320				
		.0318			
1.3	.9638		–.0099		
		.0219		0	
1.4	.9855		–.0099		0
		.0120		0	
1.5	.9975		–.0099		
		.0021			
1.6	.9996				

By Newton's forward formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + ---$$

for min value of $y, \frac{dy}{dp} = 0 \Rightarrow .0318 + \frac{(2p-1)}{2}(-.0099) = 0$

$$\Rightarrow (2p-1) = \frac{2(.0318)}{.0099} \Rightarrow p = 3.7121$$

Putting the value of p in $x = x_0 + ph$

$x = 1.2 + (.1)(3.7121) = 1.57121$

Using Newton's Backward Formula

$$p = \frac{1.57121 - 1.6}{.1} = -0.2879$$

$$y = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + ---$$

$$y = .9996 + \frac{(-.2879)(.0021)}{2} + \frac{(-.2879)(.7121)}{6}(-.0099)$$

$y = .9996$

8. Find the derivative of the function tabulated below at the point x = 1.1. **(April 2009)**

x	1.0	1.2	1.4	1.6	1.8	2.0
f(x)	0	0.128	0.544	1.296	2.432	4.00

Sol. **The forward difference table is**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	0					
		0.1280				
1.2	0.1280		0.2880			
		0.4160		0.0480		
1.4	0.5440		0.3360		0	
		0.7520		0.0480		0
1.6	1.2960		0.3840		0	
		1.1360		0.0480		
1.8	2.4320		0.4320			
		1.5680				
2.0	4.00					

Here we have $x_0 = 1$, $h = 0.2$ and $x = 1.1$

$$\therefore\ p = \frac{x - x_0}{h} = \frac{1.1-1}{0.2} = \frac{0.1}{0.2} = \frac{1}{2}$$

By Newton's forward formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots\ldots\right] \quad (1)$$

Putting $p = \frac{1}{2}$ in (1) and (2), we get

$$\left[\frac{dy}{dx}\right]_{x=1.1} = \frac{1}{0.2}\left[0.1280 + 0 + \frac{3\left(\frac{1}{4}\right) - 6\left(\frac{1}{2}\right) + 2}{6} \times (0.0480)\right]$$

$$= \frac{1}{0.2}\left[0.1280 - \frac{1}{24}(0.0480)\right]$$

$$= \frac{1}{0.2}[0.1280 - 0.0020] = 0.63$$

4

NUMERICAL INTEGRATION

1. Use Gauss quadrature formula to evaluate: $\int_5^{12} \frac{dx}{x}$. **(September 2013)**

Sol. First we transform the interval $(5, 12)$ to $(-1, 1)$

Here we have $a = 5$, $b = 12$

Now $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{7u+17}{2}$

$\therefore\ dx = \frac{7}{2}du$

$\therefore\ I = \int_{-1}^{1} \frac{2}{7u+17} \times \frac{7}{2} du$

$= 7\int_{-1}^{1} \frac{1}{7u+17} du$

$= 7\sum_{I=1}^{3} w_i g(u_i) = 7\sum_{i=1}^{3} w_i g(u\ )$

where $g(u_i) = \frac{1}{7u_i + 17}$ $(\because\ n = 3)$

Putting the values of abscissae and weights corresponding to $n = 3$ from standard table, we obtain

$$I = 7\left[(0.55555)\left(\frac{1}{17+7(0.77459)}\right) + (0.88889)\left(\frac{1}{17+7\times 0}\right)\right.$$

$$\left. + (0.55555)\left(\frac{1}{17+7(-0.77459)}\right)\right]$$

$$= 7\left[(0.55555)\left(\frac{1}{22.42213}\right) + (0.88889)\left(\frac{1}{17}\right) + (0.55555)\left(\frac{1}{11.57787}\right)\right]$$

$= 7[(0.55555)(0.44599) + (0.88889)(0.058824) + (0.55555)(0.086372)]$

$= 7[0.024777 + 0.052288 + 0.047984]$

$= 7(0.125049)$

$= 0.875343$

$=$ 0.87534 upto five decimal places.

2. Using four-point Gauss quadrature formula, evaluate $\int_{0.2}^{2.6} e^{-x} dx$. **(April 2013)**

Sol. $a = 0.2 \quad b = 2.6$

Put $x = \frac{(2.6-0.2)}{2}u + \frac{(2.6+0.2)}{2} = 1.2u + 1.4$

$dx = 1.2du$

$I = 1.2\int_{-1}^{1} e^{-(1.2u+1.4)} du$

$= 1.2\sum_{i=1}^{4} w_i g(u_i)$ where $g(u_i) = e^{-(1.2u_i+1.4)}$

$= (1.2)\left[(0.34785)\left(e^{-((1.2)(-.86114)+1.4)}\right)\right.$

$+ (0.65214)\left(e^{-((1.2)(-.33998)+1.4)}\right)$

$+ (0.34785)\left(e^{-((1.2)(.86114)+1.4)}\right)$

$\left. + (0.65214)\left(e^{-((1.2)(.33998)+1.4)}\right)\right]$

$= (1.2)[(.34785)(.69306) + (.65214)(.37082)$

$+ (.34785)(.08774) + (.65214)(.16398)]$

$= (1.2)[0.24108 + 0.24182 + 0.03052 + 0.10693]$

$= 0.7442$

3. Apply Simpson's $\frac{3}{8}$ th rule of evaluate $\int_0^1 \frac{1}{1+x} dx$ with $h = \frac{1}{6}$. **(September 2012)**

Sol. To find $I = \int_0^1 \frac{1}{1+x} dx$

Divide the range of integration (0, 1) into six equal parts each of width $h=\frac{1-0}{6}=\frac{1}{6}$

Values of the function $y=\frac{1}{1+x}$ for each subdivision are as given below:

x:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$y=\frac{1}{1+x}$:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$

Now, by Simpson's $\frac{3}{8}$ rule, we get

$$\int_0^1 \frac{1}{1+x}dx=\frac{3h}{8}\left[y_0+3(y_1+y_2+y_4+y_5)+2y_3+y_6\right]$$

$$=\frac{3}{8}\times\frac{1}{6}\left[1+3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right)+2\left(\frac{2}{3}\right)+\frac{1}{2}\right]$$

$$=\frac{1}{16}\left[1+3\left(\frac{1320+1155+924+840}{1540}\right)+\frac{4}{3}+\frac{1}{2}\right]$$

$$=\frac{1}{16}\left[1+\frac{4239}{1540}+\frac{4}{3}+\frac{1}{2}\right]$$

$$=\frac{1}{16}[1+8.2577922+1.333333+0.5]$$

$$=\frac{1}{16}(11.091126)$$

$$=0.6931954$$

4. **Deduce Simpson 1/3rd rule form Newton-Cote's formula.** **(April 2012)**

Sol. We know $\int_{x_0}^{x_n} f(x)\,dx=nh\sum_{k=0}^{n} y_k C_k^n$

Now, put n = 2 in Newton-Cote's formula, we get

$$\int_{x_0}^{x_2} f(x)\,dx=2h\sum_{k=0}^{2} y_k C_k^2=2h\left[y_0C_0^2+y_1C_1^2+y_2C_2^2\right]$$

Find $C_0^2=\frac{1}{2}\int_0^2 L_0\,du$

$$=\frac{1}{2}\int_0^2\frac{(u-1)(u-2)}{(0-1)(0-2)}du=\frac{1}{2}\cdot\frac{1}{2}\int_0^2\left(u^2-3u+2\right)du$$

$$=\frac{1}{4}\left[\frac{u^3}{3}-\frac{3u^2}{2}+2u\right]_0^2=\frac{1}{4}\left[\frac{8}{3}-\frac{12}{2}+4\right]=\frac{1}{4}\left[\frac{8}{3}-2\right]=\frac{1}{6}$$

and $C_1^2=\frac{1}{2}\int_0^2 L_1\,du=\frac{1}{2}\int_0^2\frac{u(u-2)}{1(1-2)}du=\frac{-1}{2}\int_0^2\left(u^2-2u\right)du=\frac{-1}{2}\left[\frac{u^3}{3}-u^2\right]_0^2$

$$=-\frac{1}{2}\left[\frac{8}{3}-4\right]=-\frac{1}{2}\left[-\frac{4}{3}\right]=\frac{2}{3}$$

and $C_2^2=C_0^2=\frac{1}{6}$ $\qquad\left(\because C_k^n=C_{n-k}^n\right)$

Substituting the value of C_0^2, C_1^2, C_2^2 into equation (i), we get

$$\int_{x_0}^{x_2} f(x)\,dx=2h\left[\frac{1}{6}y_0+\frac{2}{3}y_1+\frac{1}{6}y_2\right]=\frac{h}{3}[y_0+4y_1+y_2]$$

$$\therefore\ \int_{x_0}^{x_2} f(x)\,dx=\frac{h}{3}[y_0+4y_1+y_2]$$

Similarly, $\int_{x_2}^{x_4} f(x)\,dx=\frac{h}{3}[y_2+4y_3+y_4]$

....

....

$$\int_{x_{n-2}}^{x_n} f(x)\,dx=\frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]$$

Adding the above integrals, we get

$$\int_{x_0}^{x_n} f(x)\,dx=\frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+....)+2(y_2+y_4+y_6+.......)\right]$$

5. **Using Gauss quadrature formula, evaluate $\int_{-1}^{1}\left(5x^3-3x^2+2x+1\right)dx$.**

(September 2011)

Sol. Here $a=-1,\ b=1$ $\qquad\therefore x=\frac{1}{2}(b-a)u+\frac{1}{2}(b+a)=u$

$\therefore\ dx=du$

Using two point Gauss Quadrature formula

$$I=\int_{-1}^{1}\left(5u^3-3u^2+2u+1\right)du$$

$$=\sum_{i=1}^{2} w_i g(u_i)$$

Putting the values of abscissae and weights corresponding to n = 2 from standard table, we get

$$I=w_1g(u_1)+w_2g(u_2)$$

$$=(1.0)\left[5(-0.57735)^3-3(-0.57735)^2+2(-0.57735)+1\right]$$
$$+(1.0)\left[5(0.57735)^3-3(0.57735)^2+2(0.57735)+1\right]$$
$$=-6(0.57735)^2+2$$
$$=-6(0.333333)+2$$
$$=-1.9999998+2$$
$$=0.000002$$
$\therefore$ I = 0 (approximately)

6. **Solve the following integration using Gauss Quadrature formula $I=\int_0^1 x\,dx$.** **(April 2011)**

Sol. First we change the limits of integration

Here we have $a=0$, $b=1$ and we put

the values of a and b in $x=\frac{(b-a)u}{2}+\frac{b+a}{2}$

$$\therefore\ x=\frac{u}{2}+\frac{1}{2}=\frac{u+1}{2}$$
$$\therefore\ dx=\frac{1}{2}du$$
$$\therefore\ I=\frac{1}{4}\int_{-1}^{1}(u+1)\,du$$
$$=\frac{1}{4}\sum_{i=1}^{4}w_i g(u_i) \text{ where } g(u_i)=u_i+1 \qquad (\because n=4)$$

Putting the values of abscissae and weights corresponding to $n=4$ from standard table, we obtain.

$$I=\frac{1}{4}\left[(0.34785)(-0.86114+1)+(0.65214)(-0.33998+1)\right.$$
$$\left.+(0.65214)(0.33998+1)+(0.34785)(0.86114+1)\right]$$
$$=\frac{1}{4}[0.0483024+0.4304254+0.8738545+0.6473975]$$
$$=0.49999$$

7. **Evaluate: $\int_{0.1}^{1}(1+x^3)\,dx$ using Simpson's rule with step size 0.1. Compare it with exact value.** **(September 2010)**

Sol. Divide the range of integration (0.1, 1) into nine equal parts so that width of each part

$$h=\frac{1-0.1}{9}=0.1$$

The table of values of x and $y=1+x^3$ is

x:	0.1	0.2	0.3	0.4	0.5
$y=1+x^3$:	1.001	1.008	1.027	1.064	1.125
x:	0.6	0.7	0.8	0.9	1.0
$y=1+x^3$:	1.216	1.343	1.512	1.729	2.0

Since $n=9$, a multiple of 3, so we use

Simpson's three-eighth rule, we have

$$\int_{0.1}^{1}(1+x^3)\,dx=\frac{3h}{8}\left[(y_0+y_9)+3(y_1+y_2+y_4+y_5+y_7+y_8)+2(y_3+y_6)\right]$$
$$=\frac{3(0.1)}{8}\left[(1.001+2)+3(1.008+1.027+1.125+1.216\right.$$
$$\left.+1.512+1.729)+2(1.064+1.343)\right]$$
$$=\frac{0.3}{8}\left[3.001+3(7.617)+2(2.407)\right]$$
$$=\frac{1}{8}(0.3)(3.001+22.851+4.814)$$
$$=\frac{1}{8}(0.3)(30.666)$$
$$=\frac{1}{8}(9.1998)$$
$$=1.149975 \qquad (1)$$

Its exact value

$$=\int_{0.1}^{1}(1+x^3)\,dx$$
$$=\left[x+\frac{x^4}{4}\right]_{0.1}^{1}$$

$$=\left(1+\frac{1}{4}\right)-\left(0.1+\frac{(0.1)^4}{4}\right)$$

$$=\frac{5}{4}-\left(0.1+\frac{0.0001}{4}\right)$$

$$=1.25-(0.1+0.000025)$$

$$=1.149975$$

$\therefore$ From (1) and (2), we get

Exact value and approximate value are equal.

8. **Evaluate $\int_4^{5.2} \log x dx$ using Weddle's rule by taking 7 ordinates. (April 2010)**

Sol. Divide the range of integration (4, 5.2) into six (a multiple of six) equal parts each of width $h=\frac{5.2-4}{6}=0.2$

The values of x and $y=\log_e x$ are tabulated below:

x:	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$y=\log_e x$:	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

By Weddle's rule, we get

$$\int_4^{5.2}\log_e x dx=\frac{3h}{10}\left[(y_0+y_6)+5(y_1+y_5)+(y_2+y_4)+6y_3\right]$$

$$=\frac{3(0.2)}{10}\left[(1.3863+1.6487)+5(1.4351+1.6094)+(1.4816+1.5686)+6(1.5261)\right]$$

$$=(0.06)[3.035+15.2225+3.0502+9.1566]$$

$$=(0.06)(30.4643)$$

$$=1.827858$$

9. **Evaluate: $\int_4^6 (3x^2+2x)dx$ using Gaussian quadrature formula by taking $n=2$.**

(September 2009)

Sol. Given $\int_4^6 (3x^2+2x)dx$

Put $x=\left(\frac{b-a}{2}\right)u+\left(\frac{b+a}{2}\right)$

$x=u+5$

$dx=du$

$$\int_4^6 f(x)dx=\int_{-1}^{1} g(u)du=\sum_{i=1}^{2} w_i g(u_i)$$

$$\int_{-1}^{1}\left(3(u+5)^2+2(u+5)du\right)=\int_{-1}^{1}\left(3u^2+32u+85\right)du$$

$$=\left[3\left(\frac{-1}{\sqrt{3}}\right)^2+32\left(\frac{-1}{\sqrt{3}}\right)+85\right](1.0)+\left[3\left(\frac{1}{\sqrt{3}}\right)^2+32\left(\frac{1}{\sqrt{3}}\right)+85\right](1.0)$$

$$=\left[1+\frac{(-32)}{\sqrt{3}}+85\right]+\left[1+\frac{32}{\sqrt{3}}+85\right]$$

Hence $\int_4^6 (3x^2+2x)dx=172$

10. **Compute $\int_0^1 \frac{x}{x^3+10}dx$ with 9 ordinates by Simpson's $\frac{1}{3}$ rule. (April 2009)**

Sol. Given $\int_0^1 f(x)dx$ where $f(x)=\frac{x}{x^3+10}$

$h=\frac{b-a}{n}=\frac{1}{9}$

x	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	1
$f(x)$	0	0.0111	0.0221	0.0332	0.044	0.0546	0.0647	0.0742	0.083	0.0909

Now Simpson $\frac{1}{3}$ Rule is

$$\int_a^b f(x)dx=\frac{h}{3}\left[(y_0+y_9)+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7)\right]$$

$$\therefore \int_0^1 \frac{x}{x^3+10}dx=\frac{1}{(9)(3)}\left[(0+0.0909)+2(0.0221+0.044+0.0647+0.083)\right.$$

$$\left.+4(0.0111+0.0332+0.0546+0.0742)\right]$$

$$=\frac{1}{27}\left[(0.0909)+(0.4276)+(0.6924)\right]$$

$$=\frac{1}{27}(1.2109)=0.0448$$

$$\therefore \int_0^1 \frac{x}{x^3+10}dx=0.0448$$

5

SYSTEM OF LINEAR EQUATION

1. Solve the following system of equations by Gauss elimination method.

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

(September 2013)

Sol. Given system is

$2x_1 - x_2 + x_3 = -1$ (1)

$2x_2 + x_3 + x_4 = 1$ (2)

$x_1 + 2x_3 + x_4 = -1$ (3)

$x_1 + 2x_2 + 2x_4 = 3$ (4)

To eliminate x_1, operate $(3) - \frac{1}{2}\times(1), (4) - \frac{1}{2}\times(1)$, we get

$2x_1 - x_2 + x_3 = -1$ (5)

$2x_2 - x_3 + x_4 = 1$ (6)

$\frac{1}{2}x_2 + \frac{3}{2}x_3 + x_4 = -\frac{1}{2}$ (7)

$\frac{5}{2}x_2 - \frac{1}{2}x_3 + 2x_4 = \frac{7}{2}$ (8)

To eliminate x_2, opeate $(7) - \frac{1}{4}\times(6)$, $(8) - \frac{5}{4}\times(6)$

$2x_1 - x_2 + x_3 = -1$ (9)

$2x_2 + x_3 + x_4 = 1$ (10)

$\frac{5}{4}x_3 + \frac{3}{4}x_4 = \frac{-3}{4}$ (11)

$-\frac{7}{4}x_3 + \frac{3}{4}x_4 = \frac{9}{4}$ (12)

To eliminate x_3, operate $(12) + \frac{7}{5}\times(11)$

$2x_1 - x_2 + x_3 = -1$ (13)

$2x_2 + x_3 + x_4 = 1$ (14)

$\frac{5}{4}x_3 + \frac{3}{4}x_4 = \frac{-3}{4}$ (15)

$\frac{9}{5}x_4 = \frac{24}{20}$ (16)

From (16), $x_4 = \frac{2}{3}$

From (15), $x_3 = -1$

From (14), $x_2 = \frac{2}{3}$

From (13), $x_1 = \frac{1}{3}$

Hence the solution is

$x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = -1$, $x_4 = \frac{2}{3}$

2. Solve the following system of equations using LU decomposition method:

$\mathbf{2x - 3y + 10z = 3}$

$\mathbf{-x + 4y + 2z = 20}$

$\mathbf{5x + 2y + z = -12}$. **(April 2013)**

Sol. Given equations can be written in the matrix from as AX = B (1)

Where $A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, Y = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$

Let A = LU (2)

where

$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

(i) Comparing the elements of first column

$l_{11} = 2,\ l_{21} = -1,\ l_{31} = 5$

(ii) Comparing the elements of first row

$l_{11}u_{12} = -3,\ l_{11}u_{13} = 10$

$\therefore\ u_{12} = \frac{-3}{2},\quad u_{13} = 5$

(iii) Comparing the elements of second column

$l_{21}u_{12} + l_{22} = 4 \qquad \Rightarrow (-1)\times\left(\frac{-3}{2}\right) + l_{22} = 4$

$\therefore\ l_{22} = 4 - \frac{3}{2} = \frac{5}{2}$

$l_{31}u_{12} + l_{32} = 2 \qquad \Rightarrow 5\times\left(\frac{-3}{2}\right) + l_{32} = 2$

$\therefore\ l_{32} = 2 + \frac{15}{2} = \frac{19}{2}$

(iv) Comparing the elements of second row

$l_{21}u_{13} + l_{22}u_{23} = 2$

$(-1)\times 5 + \frac{5}{2}u_{23} = 2$

$\therefore\ u_{23} = \frac{2}{5}(2+5) = \frac{14}{5}$

(v) Comparing the elements of third column.

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1$

$5\times 5 + \frac{19}{2}\times\frac{14}{5} + l_{33} = 1$

$\therefore\ l_{33} = 1 - 25 - \frac{133}{5} = \frac{-253}{5}$

$$\therefore\ L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 0 \\ 5 & \frac{19}{2} & \frac{-253}{5} \end{bmatrix}$$

$$\text{and } U = \begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

From (1) and (2), we have (LU) X = B

or L(UX) = B

Putting UX = Y

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, we get

LY = B

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 0 \\ 5 & \frac{19}{2} & \frac{-253}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

$\therefore\ 2y_1 = 3 \qquad \Rightarrow y_1 = \frac{3}{2}$

$-y_1 + \frac{5}{2}y_2 = 20 \quad \Rightarrow y_2 = \frac{2}{5}\left(20 + \frac{3}{2}\right) = \frac{43}{5}$

$5y_1 + \frac{19}{2}y_2 - \frac{253}{5}y_3 = -12$

$\Rightarrow\ y_3 = \frac{5}{253}\left(12 + 5\times\frac{3}{2} + \frac{19}{2}\times\frac{43}{5}\right)$

$= \frac{5}{253}\left(\frac{120 + 75 + 817}{10}\right)$

$$=\frac{5}{253}\times\frac{1012}{10}=2$$

From (3), U X = Y

$$\begin{bmatrix}1 & \frac{-3}{2} & 5\\ 0 & 1 & \frac{14}{5}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\ \frac{43}{5}\\ 2\end{bmatrix}$$

$$\Rightarrow x-\frac{3}{2}y+5x=\frac{3}{2} \quad \textbf{(4)}$$

$$y+\frac{14}{5}z=\frac{43}{5} \quad \textbf{(5)}$$

$$z=2 \quad \textbf{(6)}$$

From (5), $y=\frac{43}{5}-\frac{14}{5}z$

$$=\frac{43}{5}-\frac{28}{5}=3$$

From (4), $x=\frac{3}{2}+\frac{3}{2}y-5z$

$$=\frac{3}{2}+\frac{3}{2}\times3-5\times2$$

$$=6-10=-4$$

Hence the solution is $x=-4,\ y=3,\ z=2$

3. **Using Cholesky decomposition method, solve the equations:**
x + y + z = 3
x + 2y + 3z = 6
x + 3y + 6z = 10. **(September 2012)**

Sol. Given equations can be written in matrix from as AX = B

Where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}3\\6\\10\end{bmatrix}$

Write $A=LL'$

Where $L=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$ and $L'=\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}$

$\therefore$ From (1), $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_1 & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}$

$\Rightarrow l_{11}^2=1 \therefore l_{11}=1$

$l_{11}l_{21}=1 \quad \therefore l_{21}=1$

$l_{11}l_{31}=1 \quad \therefore l_{31}=1$

$l_{21}^2+l_{22}^2=2 \therefore l_{22}^2=2 \therefore l_{22}=1$

$l_{21}l_{31}+l_{22}l_{32}=3$

$l_{32}=\frac{3-1\times1}{1}=2$

$l_{31}^2+l_{32}^2+l_{33}^2=6 \therefore l_{33}^2=6-1^2-2^2=1$

$\therefore l_{33}=1$

$\therefore L=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}$ and $L'=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$

From (1) and (2), we have

$(LL')X=B$

$\Rightarrow L(L'X)=B$

Putting $L'X=Y$

where

$Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$, we have

LY = B

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}3\\6\\10\end{bmatrix}$$

$\therefore y_1=3$

$y_1+y_2=6 \qquad \Rightarrow y_2=3$

$y_1+2y_2+y_3=10 \Rightarrow y_3=10-3-6=1$

From (3), $L'X = Y$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore\ x + y + z = 3$

$y + 2z = 3$

$z = 1$

From (5),

$y = 3 - 2z = 3 - 2 = 1$

From (4),

$x = 3 - y - z = 3 - 1 - 1 = 1$

Hence the solution is $x = 1,\ y = 1,\ z = 1$

4. **Solve:** $x + \frac{y}{2} + \frac{z}{3} = 1$ **(September 2012)**

$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 0$

$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 0$, **by Gauss's elimination method.**

Sol. Given system is $x + \frac{1}{2}y + \frac{1}{3}z = 1$ (1)

$\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 0$ (2)

$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 0$ (3)

To eliminate x, operate $(2) - \frac{1}{2}\times(1), (3) - \frac{1}{3}\times(1)$, we get

$\frac{1}{12}y + \frac{1}{12}z = -\frac{1}{2}$ (4)

$\frac{1}{12}y + \frac{4}{45}z = -\frac{1}{3}$ (5)

To eliminate y, operate (5) – (4), we get

$\frac{1}{180}z = \frac{1}{6}$

$\therefore\ z = 30$

Putting the value of z in (4), we get

$\frac{1}{12}y + \frac{30}{12} = -\frac{1}{2}$

$\Rightarrow \frac{1}{12}y = -\frac{30}{12} - \frac{1}{2} = -\frac{36}{12}$

$\therefore\ y = -36$

Putting the value of y and z in (1), we get

$x - \frac{36}{2} + \frac{30}{3} = 1$

$\therefore\ x = 1 + 18 - 10 = 9$

Hence the solution is $x = 9,\ y = -36$ and $z = 30$

5. **Apply Gauss-Siedel iterative method to solve the following system of linear equations: $27x + 6y - z = 85,\ 6x + 15y + 2z = 72,\ x + y + 54z = 110$.**

(April 2012)

Sol. Given equations can be written as

$$\left.\begin{aligned} x &= \frac{1}{27}(85 - 6y + z) \\ y &= \frac{1}{15}(72 - 6x - 2z) \\ z &= \frac{1}{54}(110 - x - y) \end{aligned}\right]$$

Starting with initial approximation $y = 0,\quad z = 0$

Iteration 1. Putting $y = 0,\quad z = 0$ in the right side of first equation of (1), we get

$x^{(1)} = \frac{1}{27}(85 - 0 + 0) = 3.15$

Putting $x = 3.15,\quad z = 0$ in the second equation, we get

$y^{(1)} = \frac{1}{15}(72 - 6\times 3.15 - 0) = 3.54$

Putting $x = 3.15,\quad y = 3.54$ in the third equation, we get

$z^{(1)} = \frac{1}{54}(110 - 3.15 - 3.54) = 1.91$

Iteration 2. Putting $y = 3.54,\quad z = 1.91$ in the first equation, we get

$x^{(2)} = \frac{1}{27}(85 - 6\times 3.54 + 1.91)$

$= \frac{1}{27}(85 - 21.24 + 1.91) = 2.43$

Putting $x=2.43,\quad z=1.91$ in the second equation, we get

$$y^{(2)}=\frac{1}{15}(72-6\times 2.43-2\times 1.91)$$

$$=\frac{1}{15}(72-14.58-3.82)=3.57$$

Putting $x=2.43,\quad y=3.57$ in the third equation, we get

$$z^{(2)}=\frac{1}{54}(110-2.43-3.57)=1.926$$

Iteration 3. Putting $y=3.57,\quad z=1.926$ in first equation, we get

$$x^{(3)}=\frac{1}{27}(85-6\times 3.57+1.926)=2.426$$

Putting $x=2.426,\quad z=1.926$ in second equation, we get

$$y^{(3)}=\frac{1}{15}(72-6\times 2.426-2\times 1.926)=3.572$$

Putting $x=2.426,\quad z=3.572$ in third equation, we get

$$z^{(3)}=\frac{1}{54}(110-2.426-3.572)=1.926$$

We observe that the values obtained in second and third iterations are sufficiently close.

Hence $x=2.426,\quad y=3.572,\quad z=1.926$ is the required solution.

6. **Solve by using Gauss's elimination method, the following system equations:** **$2x+6y+z=3,\ 3x+2y-2z=2,\ x-y+z=6$.** **(April 2012)**

Sol. $2x+6y+z=3$ (1)

$3x+2y-2z=2$ (2)

$x-y+z=6$ (3)

To eliminate x, operate $(2)-\frac{3}{2}(1),\ (3)-\frac{1}{2}(1)$

$2x+6y+z=3$

$-7y-\frac{7}{2}z=\frac{-5}{2}$ (4)

$-4y+\frac{z}{2}=\frac{9}{2}$ (5)

To eliminate y, operate $(5)-\frac{4}{7}(4)$

$2x+6y+z=3$

$-7y-\frac{7}{2}z=\frac{-5}{2}$

$\frac{5}{2}z=\frac{83}{14}$ (6)

From (6), $z=\frac{83}{35}$

From (4), $y=\frac{1}{7}\left(\frac{5}{2}-\frac{7}{2}\left(\frac{83}{35}\right)\right)=\frac{-58}{70}=\frac{-29}{35}$

From (1), $x=\frac{1}{2}\left(3-6\left(\frac{-58}{70}\right)-\frac{83}{35}\right)=\frac{14}{5}$

Hence

$$x=\frac{14}{5},\quad y=\frac{-29}{35},\quad z=\frac{83}{35}$$

7. **Solve the system of linear equations**
$2x+4y-6z=-8$
$x+3y+z=10$
$2x-4y-2z=-1$ by Gauss elimination method. **(September 2011)**

Sol. $2x+4y-6z=-8$ (1)

$x+3y+z=10$ (2)

$2x-4y-2z=-1$ (3)

To eliminate x, operate $(2)-\frac{1}{2}\times(1),(3)-(1)$

$2x+4y-6z=-8$

$y+4z=14$ (4)

$-8y+4z=7$ (5)

To eliminate y, (5) + 8(4)

$2x+4y-6z=-8$

$y+4z=14$

$36z=119$ (6)

$z=\frac{119}{36}$

From (4), $y=14-4\left(\frac{119}{36}\right)=\frac{7}{9}$

From (1), $x=\frac{1}{2}\left(-8-4\left(\frac{7}{9}\right)+6\left(\frac{119}{36}\right)\right)=\frac{157}{36}$

$x=\frac{157}{36}, \quad y=\frac{7}{9}, \quad z=\frac{119}{36}$

8. **Solve the system of linear equations** **(September 2011)**

$5x+2y+z=12$

$x+4y+2z=15$

$x+2y+5z=20$ by Gauss-Seidel method.

Sol. The given system of equations can be written as

$$x=\frac{1}{5}(12-2y-z)$$

$$y=\frac{1}{4}(12-x-2z)$$

$$z=\frac{1}{5}(20-x-2y) \qquad \textbf{(1)}$$

Take the initial approximations as $y=0, z=0$

Iteration 1. Putting $y=0$, $z=0$ in right side of the first equation of (1), we get

$$x^{(1)}=\frac{1}{5}(12-0-0)=2.4$$

Putting $x=2.4$, $z=0$ in the second equation, we get

$$y^{(1)}=\frac{1}{4}(15-2.4-0)=\frac{12.6}{4}=3.15$$

Putting $x=2.4$, $y=3.15$ in the third equation, we get

$$z^{(1)}=\frac{1}{5}(20-2.4-2\times3.15)=\frac{11.3}{5}=2.26$$

Iteration 2. Putting $y=3.15$, $z=2.26$ in the first equation, we get

$$x^{(2)}=\frac{1}{5}(12-2\times3.15-2.26)=0.688$$

Putting $x=0.688$, $z=2.26$ in the second equation, we get

$$y^{(2)}=\frac{1}{4}(15-0.688-2\times2.26)=2.448$$

Putting $x=0.688$, $y=2.448$ in the third equation, we get

$$z^{(2)}=\frac{1}{5}(20-0.688-2\times2.448)=2.8832$$

Iteration 3. Putting $y=2.448$, $z=2.8832$ in the first equation, we get

$$x^{(3)}=\frac{1}{5}(12-2\times2.448-2.8832)=0.84416$$

Putting $x=0.84416$, $z=2.8832$ in the second equation, we get

$$y^{(3)}=\frac{1}{4}(15-0.84416-2\times2.8832)=2.09736$$

Putting $x=0.84416$, $y=2.09736$ in the third equation, we get

$$z^{(3)}=\frac{1}{5}(20-0.688-2\times2.09736)=2.99222$$

Iteration 4. Putting $y=2.09736$, $z=2.99222$

$$x^{(4)}=\frac{1}{5}(12-2\times2.0936-2.99222)=0.962612$$

Putting $x=0.962612$, $z=2.99222$ in the second equation, we get

$$y^{(4)}=\frac{1}{4}(15-0.962612-2\times2.99222)=2.013237$$

Putting $x=0.0962612$, $y=2.013237$ in the third equation, we get

$$z^{(4)}=\frac{1}{5}(20-0.0962612-2\times2.013237)=3.0021828$$

Iteration 5. Putting $y=2.013237$, $z=3.0021828$ in the first equation, we get

$$x^{(5)}=\frac{1}{5}(12-2\times2.0132337-3.0021828)=0.99426864$$

Putting $x=0.99426864$, $z=3.0021828$ in the second equation, we get

$$y^{(5)}=\frac{1}{4}(15-0.99426864-2\times3.0021828)=2.00034144$$

Putting $x=0.99426864$, $y=2.00034144$ in the third equation, we get

$$z^{(5)}=\frac{1}{5}(20-0.99426864-2\times2.00034144)=3.00109696$$

The values in the fourth and fifth iterations being practically the same, we can stop. Hence the solution is $x=1, y=2, z=3$

9. **Solve the system of equations:** **(April 2011)**

$5x-2y+z=4$

$7x+y-5z=8$

$3x+7y+4z=10$ Using LU decomposition.

Sol. Given equations can be written in matrix form AX = B where $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Let A = LU where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Comparing $l_{11} = 5$, $l_{21} = 7$, $l_{31} = 3$

$u_{12} = \frac{-2}{5}$, $u_{13} = \frac{1}{5}$, $u_{22} = \frac{19}{5}$

$u_{23} = \frac{-32}{19}$, $l_{32} = \frac{41}{5}$, $l_{33} = \frac{327}{19}$

Now L(UX) = B

Putting UX = Y where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

LY = B

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$\Rightarrow 5y_1 = 4 \qquad \Rightarrow y_1 = \frac{4}{5}$

Also $7y_1 + \frac{19}{5}y_2 = 8 \Rightarrow y_2 = \frac{12}{19}$

Also, $3y_1 + \frac{41}{5}y_2 + \frac{327}{19}y_3 = 10$

$\Rightarrow y_3 = \frac{46}{327}$

Now UX = Y

$$\begin{bmatrix} 1 & \frac{-2}{5} & \frac{1}{5} \\ 0 & 1 & \frac{-32}{19} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{12}{19} \\ \frac{46}{327} \end{bmatrix}$$

$\Rightarrow z = \frac{46}{327}$

Also $y - \frac{32}{19}\left(\frac{46}{327}\right) = \frac{12}{19}$

$\Rightarrow y = \frac{284}{327}$

Also $x - \frac{2}{5}\left(\frac{284}{327}\right) + \frac{1}{5}\left(\frac{46}{327}\right) = \frac{4}{5}$

$\Rightarrow x = \frac{122}{109}$

Hence the solution is

$x = \frac{122}{109}$, $y = \frac{284}{327}$, $z = \frac{46}{327}$

10. Solve using LU decomposition:

$\mathbf{2x + 3y + z = 9}$

$\mathbf{x + 2y + 3z = 6}$

$\mathbf{3x + y + 2z = 8}$ **(September 2010, September 2009)**

Sol. Given equations can be written as AX = B where

$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Let A = L U where

$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

(i) Comparing the elements of first column

$l_{11} = 2,\ l_{21} = 1,\ l_{31} = 3$

(ii) Comparing the elements of first row

$l_{11}u_{12} = 3,\ l_{11}u_{13} = 1$

$\therefore\ u_{12} = \frac{3}{2},\ u_{13} = \frac{1}{2}$

(iii) Comparing the elements of second column

$l_{21}u_{12} + l_{22} = 2$ or $1 \times \frac{3}{2} + l_{22} = 2$

$\therefore\ l_{22} = 2 - \frac{3}{2} = \frac{1}{2}$

$l_{31}u_{12} + l_{32} = 1$ or $3 \times \frac{3}{2} + l_{32} = 1$

$\therefore\ l_{32} = 1 - \frac{9}{2} = \frac{-7}{2}$

(iv) Comparing the elements of second row

$l_{21}u_{13} + l_{23} = 3$

$1 \times \frac{1}{2} - \left(\frac{1}{2}\right) u_{23} = 3$

$\therefore\ u_{23} = 5$

(v) Comparing the elements of third column

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$

$3 \times \frac{1}{2} + \left(\frac{-7}{2}\right) \times 5 + l_{33} = 2$

$\therefore\ l_{33} = 2 - \frac{3}{2} + \frac{35}{2} = 18$

$$\therefore\ L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 3 & -\frac{7}{2} & 18 \end{bmatrix} \text{ and}$$

$$U = \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

From (1) and (2), we have

(L U) X = B

or L(U X) = B

Putting U X = Y where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, we get

L Y = B

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 3 & \frac{-7}{2} & 18 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$\Rightarrow 2y_1 = 9 \qquad \therefore\ y_1 = \frac{9}{2}$

$y_1 + \frac{y_2}{2} = 6 \qquad \therefore\ y_2 = 2(6 - y_1) = 2\left(6 - \frac{9}{2}\right) = 3$

$3y_1 - \frac{7}{2}y_2 + 18y_3 = 8 \qquad \therefore\ y_3 = \frac{1}{18}\left(8 - 3y_1 + \frac{7}{2}y_2\right)$

or $y_3 = \frac{1}{18}\left(8 - 3 \times \frac{9}{2} + \frac{7}{2} \times 3\right) = \frac{5}{18}$

From (3), UX = Y

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ 3 \\ \frac{5}{18} \end{bmatrix}$$

$$x+\frac{3}{2}y+\frac{z}{2}=\frac{9}{2} \qquad (4)$$

$$y+5z=3 \qquad (5)$$

$$z=\frac{5}{18} \qquad (6)$$

From (5), $y+5\times\frac{5}{18}=3$

or $y=3-\frac{25}{18}=\frac{29}{18}$

From (4), $x+\frac{3}{2}\times\frac{29}{18}+\frac{1}{2}\times\frac{5}{18}=\frac{9}{2}$

$$x=\frac{9}{2}-\frac{29}{12}-\frac{5}{36}=\frac{35}{18}$$

Hence the solution is $x=\frac{35}{18}$, $y=\frac{29}{18}$ and $z=\frac{5}{18}$

11. Solve by Jacobi's iteration method, the system of equations:

$5x-y+z=10$

$2x+4y=12$

$x+y+5z=-1$ **(April 2010)**

Sol. The given equations can be written as

$$\left.\begin{aligned} x&=\frac{1}{5}(10+y-z) \\ y&=\frac{1}{4}(12-2x) \\ z&=\frac{1}{5}(-1-x-y) \end{aligned}\right] \qquad (1)$$

Let the initial approximations be $x=2,\ y=3,\ z=0$

Iteration 1. Putting the initial values in right side of (1)

$$x^{(1)}=\frac{1}{5}(10+3-0)=2.6$$

$$y^{(2)}=\frac{1}{4}(12-2\times2)=2.0$$

$$z^{(1)}=\frac{1}{5}(-1-2-3)=-1.2$$

Iteration 2. Putting the values obtained in the first iteration in right side of (1), we get

$$x^{(2)}=\frac{1}{5}(10+2.0+1.2)=2.64$$

$$y^{(2)}=\frac{1}{4}(12-2\times2.6)=1.7$$

$$z^{(2)}=\frac{1}{5}(-1-2.6-2.0)=-1.12$$

Iteration 3. Putting the values obtained in the second iteration in right side of (1), we get

$$x^{(3)}=\frac{1}{5}(10+1.7+1.12)=2.564$$

$$y^{(3)}=\frac{1}{4}(12-2\times2.64)=1.68$$

$$z^{(3)}=\frac{1}{5}(-1-2.64-1.7)=-1.068$$

Iteration 4. Putting the values obtained in the third iteration in right side of (1), we get

$$x^{(4)}=\frac{1}{5}(10+1.68+1.068)=2.5496$$

$$y^{(4)}=\frac{1}{4}(12-2\times2.564)=1.718$$

$$z^{(4)}=\frac{1}{5}(-1-2.564-1.68)=-1.0488$$

Iteration 5. Putting the values obtained in the fourth iteration in right side of (1), we get

$$x^{(5)}=\frac{1}{5}(10+1.718+1.0488)=2.553$$

$$y^{(5)}=\frac{1}{4}(12-2\times2.5496)=1.725$$

$$z^{(5)}=\frac{1}{5}(-1-2.5496-1.718)=-1.054$$

Iteration 6. Putting the values obtained in the fifth iteration in right side of (1), we get

$$x^{(6)}=\frac{1}{5}(10+1.725+1.054)=2.556$$

$$y^{(6)} = \frac{1}{4}(12 - 2\times 2.553) = 1.724$$

$$z^{(6)} = \frac{1}{5}(-1 - 2.553 - 1.725) = -1.056$$

The values in the fifth and six iteration are being practically the same, we can stop.

$\therefore$ The solution is $x = 2.556$, $y = 1.724$ and $z = -1.056$

12. Solve the following system of linear equations by Gausselimination method

2x + y + z = 10

3x + 2y + 3x = 18

x + 4y + 9z = 16 **(April 2009)**

Sol. Given

$x + 4y + 9z = 16$ (1)

$2x + y + z = 10$ (2)

$3x + 2y + 3z = 18$ (3)

To eliminate x, operate (2) – 2 × (1) and (3) – 3 × (1), we get

$-7y - 17z = -22$ (4)

$-10y - 24z = -30$ (5)

To eliminate y, operate $(5) - \frac{10}{7}\times(4)$, we get

$$\frac{2z}{7} = \frac{10}{7}$$

$\therefore\ z = 5$

Putting the value of z in (4), we get

$-7y - 85 = -22$

$-7y = 85 - 22$

$-7y = 63$

$y = -9$

Putting the values of y and z in (1), we get

$x - 36 + 45 = 16$

$x = 16 + 36 - 45 = 7$

Hence the solution is $x = 7$, $y = -9$ and $z = 5$

6

ALGEBRAIC EIGEN VALUE PROBLEMS

1. Reduce the matrix: **(September 2013)**

$\begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ **to tri-diagonal form using House-holder's method.**

Sol. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$\therefore\ a_{11} = 1,\ a_{12} = 4,\ a_{13} = 3,\ a_{22} = 1,\ a_{23} = 2,\ a_{33} = 1$

We have $S = \sqrt{a_{12}^2 + a_{13}^2} = \sqrt{16+9} = 5$

Take $V = \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}$

Now $v_2{}^2 = \frac{1}{2}\left[1 \mp \frac{a_{12}}{S}\right] = \frac{1}{2}\left[1 + \frac{4}{5}\right] = \frac{9}{10}$

the positive sign being chosen, since a_{12} is positive

$\Rightarrow v_2 = \frac{3}{\sqrt{10}}$

and $v_3 = \pm\frac{a_{13}}{2v_2\sqrt{a_{12}^2 + a_{13}^2}} = \frac{3}{2\times\frac{3}{\sqrt{10}}\times 5} = \frac{1}{\sqrt{10}}$

$$\therefore V = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

Now $P = I - 2VV'$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\begin{bmatrix} 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{9}{10} & \frac{3}{10} \\ 0 & \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

Hence we have

$A_1 = PAP$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 \\ -5 & -2 & -\frac{11}{5} \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 0 \\ -5 & \frac{73}{25} & -\frac{14}{25} \\ 0 & -\frac{14}{25} & -\frac{11}{25} \end{bmatrix}$$

which is the required tri-diagonal form

2. **Using Power method, find the largest eigen value and the corresponding eigen vector of the matrix** $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. **(April 2013)**

Sol. Let the initial eigen vector be $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

then $A X^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1X^{(1)}$

The first approximation to the eigen value is 1 and the corresponding eigen vector is

$X^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Hence, we have $A X^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = 7X^{(2)}$

$\therefore$ The second approximation to the eigen value is 7 and the corresponding eigen vector

is $X^{(2)} = \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix}$

$$A\ X^{(2)}=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.4\\0\end{bmatrix}=\begin{bmatrix}3.4\\1.8\\0\end{bmatrix}=(3.4)\begin{bmatrix}1\\0.52\\0\end{bmatrix}=3.4X^{(3)}$$

$$A\ X^{(3)}=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.52\\0\end{bmatrix}=\begin{bmatrix}4.12\\2.04\\0\end{bmatrix}=(4.12)\begin{bmatrix}1\\0.49\\0\end{bmatrix}=4.12\ X^{(4)}$$

$$A\ X^{(4)}=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.49\\0\end{bmatrix}=\begin{bmatrix}3.94\\1.98\\0\end{bmatrix}=(3.94)\begin{bmatrix}1\\0.50\\0\end{bmatrix}=3.94\ X^{(5)}$$

$$A\ X^{(5)}=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.50\\0\end{bmatrix}=\begin{bmatrix}4\\2\\0\end{bmatrix}=4\begin{bmatrix}1\\0.5\\0\end{bmatrix}=4\ X^{(6)}$$

$\because\ X^{(6)}=X^{(5)}$

Hence the largest eigen value is 4 and the corresponding eigen vector is $\begin{bmatrix}1\\0.5\\0\end{bmatrix}$

3. **Using Jacobi's Method find all the eigen values and eigen vectors of the matrix:**

$$A=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}$$

(September 2012)

Sol. The largest non-diagonal element is $a_{13}=a_{31}=1>0$

Since $a_{11}=a_{33}=5$

$$\therefore\ \tan 2\theta=\frac{2a_{13}}{a_{11}-a_{33}}=\frac{2\times 1}{5-5}=\infty$$

$$\therefore\ 2\theta=\frac{\pi}{2}\qquad\Rightarrow\theta=\frac{\pi}{4}$$

The transformation matrix P_1 is $\begin{bmatrix}\cos\theta&0&-\sin\theta\\0&1&0\\\sin\theta&0&\cos\theta\end{bmatrix}$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\therefore\ P_1^{-1}=P_1'=\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{bmatrix}$$

$(\because\ P_1$ is orthogonal matrix $\therefore\ P_1^{-1}=P_1')$

$\therefore$ the transformation gives

$A_1=P_1AP_1^{-1}$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\0&1&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}4&0&0\\0&-2&0\\0&0&6\end{bmatrix}$$, which is a diagonal matrix.

Hence the eigen value of the given matrix A are 4, –2, 6.

Now the eigen vectors of A are the columns of the matrix P_1^{-1}

$\therefore$ The eigen vectors corresponding to the eigen values 4, –2, 6 respectively are

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

4. Compute eigen values and the corresponding eigen vectors of the matrix $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ using Jacobi method.

(April 2012)

Sol. The largest off diagonal element is

$a_{12} = a_{21} = 2$

$$\therefore \tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2 \times 2}{(-2)-(-2)} = \frac{4}{0} = \infty$$

$$\therefore 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Transformation Matrix $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now $PAP^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}$ which is diagonal matrix

$\therefore$ eigen values of matrix A are 0, –4

5. Using Given's method, reduce the matrix $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 3 \end{pmatrix}$ to tridiagonal form.

(September 2011)

Sol. Given matrix has only one non-tridiagonal element $a_{13} = (3)$ which has to be reduced to zero. Hence only one rotation is required.

To annihilate a_{13}, we define the orthogonal matrix in the plane (2, 3) as

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where θ is given by, $\tan\theta = \frac{a_{13}}{a_{12}} = \frac{3}{1} = 3$

$\therefore \sec^2\theta = 1 + \tan^2\theta = 1 + 9 = 10$

$\therefore \sec^2\theta = \sqrt{10} \Rightarrow \cos\theta = \frac{1}{\sqrt{10}}$

$\therefore \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{10} = \frac{9}{10}$

$\Rightarrow \sin\theta = \frac{3}{\sqrt{10}}$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

So that $P^{-1} = P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ 0 & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$

$$\therefore P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ 0 & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ \frac{10}{\sqrt{10}} & \frac{10}{\sqrt{10}} & \frac{11}{\sqrt{10}} \\ 0 & -\frac{10}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \sqrt{10} & 0 \\ \sqrt{10} & 4.3 & -1.9 \\ 0 & -1.9 & 2.7 \end{bmatrix} = \begin{bmatrix} 2 & 3.16 & 0 \\ 3.16 & 4.3 & -1.9 \\ 0 & -1.9 & 2.7 \end{bmatrix}$$

which is the required tri-diagonal matrix.

6. **Find all the Eigen values of the matrix A =** $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ **Using the Jacobi method.**

(April 2011)

Sol. Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Largest off diagonal element is $a_{12} = 2$

$$\tan\theta = \frac{2a_{12}}{a_{11}-a_{22}} = \frac{2(2)}{1-1} = \frac{4}{\infty} \Rightarrow \theta = \frac{\pi}{4}$$

Transformation matrix $P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A_1 = P_1^{-} AP_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 1 \end{bmatrix}$$

Largest off diagonal element is $a_{23} = \frac{3}{\sqrt{2}}$

$$\tan 2\theta = \frac{2a_{23}}{a_{22}-a_{33}} = \frac{3\sqrt{2}}{-2} = \frac{-3}{\sqrt{2}} \Rightarrow \theta = -0.56514$$

Transformation matrix $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .84451 & .53553 \\ 0 & -.53553 & .84451 \end{bmatrix}$

$$A_2 = P_2^{-1} A_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .84451 & -.53553 \\ 0 & .53553 & .84451 \end{bmatrix} \begin{bmatrix} 3 & 0 & .7071 \\ 0 & -1 & 2.12132 \\ .7071 & 2.12132 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & .84451 & .53553 \\ 0 & -.53553 & .84451 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -.37867 & .59715 \\ -.37867 & -2.34514 & 0 \\ .59715 & 0 & 2.34514 \end{bmatrix}$$

Largest off diagonal dement $a_{13} = .59715$

$$\tan 2\theta = \frac{2a_{13}}{a_{11}-a_{33}} = \frac{2(.59715)}{3-2.34514} \Rightarrow \theta = .53462$$

$$P_3 = \begin{bmatrix} .86046 & 0 & -.50951 \\ 0 & 1 & 0 \\ .50951 & 0 & .86046 \end{bmatrix}$$

$$A_3 = P_3^{-1} A_2 P_3 = \begin{bmatrix} .86046 & 0 & .50951 \\ 0 & 1 & 0 \\ -.50951 & 0 & .86046 \end{bmatrix} \begin{bmatrix} 3 & -.37867 & .59715 \\ -.37867 & -2.34514 & 0 \\ .59715 & 0 & 2.34514 \end{bmatrix}$$

$$\begin{bmatrix} .86046 & 0 & -.50951 \\ 0 & 1 & 0 \\ .50951 & 0 & .86046 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3.35356 & -0.32583 & 0 \\ -0.32583 & -2.34514 & .19293 \\ 0 & .19293 & 2.03737 \end{bmatrix}$$

Largest off diagonal element $a_{12} = -.32583$

$$\tan 2\theta = \frac{2(a_{12})}{a_{11}-a_{22}} = \frac{2(-0.32583)}{(3.35356+2.34514)} \Rightarrow \theta = -0.05692$$

$$P_4 = \begin{bmatrix} .99838 & .05688 & 0 \\ -0.05688 & .99838 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = P_4^{-1} A_3 P_4 = \begin{bmatrix} .99838 & -.05688 & 0 \\ 0.05688 & .99838 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.35356 & -.32583 & 0 \\ -.32583 & -2.34514 & .19293 \\ 0 & .19293 & 2.0737 \end{bmatrix} \begin{bmatrix} .99838 & .05688 & 0 \\ -0.05688 & .99838 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 3.37206 & 0 & -0.01097 \\ 0 & -2.3637 & .19261 \\ -0.01097 & .19261 & 2.03737 \end{bmatrix}$$

Largest off diagonal element $a_{23} = .19261$

$$\tan 2\theta = \frac{2a_{23}}{a_{23} - a_{33}} = \frac{2(.19261)}{(-2.3637 - 2.03737)} \Rightarrow \theta = -0.04364$$

$$P_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .99904 & .04362 \\ 0 & -.04362 & .99904 \end{bmatrix}$$

$$A_5 = P_5^{-1} A_4 P_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .99904 & -.04362 \\ 0 & .04362 & .99904 \end{bmatrix} \begin{bmatrix} 3.37206 & 0 & -.01097 \\ 0 & -2.3637 & .19261 \\ -0.01097 & .19261 & 2.03737 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & .99904 & .04362 \\ 0 & -.04362 & .99904 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 3.37206 & 0 & -0.01097 \\ 0 & -2.3384 & .38365 \\ -0.01097 & .38365 & 2.0121 \end{bmatrix}$$

We finally take eigen values

(3.37206, –2.3384, 2.0121)

7. Find the largest or dominant eigen value of matrix $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by power method.

(September 2010)

Sol. Let the initial eigen vector be $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

then $AX^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1.X^{(1)}$

The first approximation to the eigen value is 1 and the corresponding eigen vector is

$X^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Hence, we have $AX^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = 7.X^{(2)}$

∴ The second approximation to the eigen value is 7 and the corresponding eigen vector

is $X^{(2)} = \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix}$

$$AX^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 1.8 \\ 0 \end{bmatrix} = (3.4) \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.4X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = (4.12) \begin{bmatrix} 1 \\ 0.49 \\ 0 \end{bmatrix} = 4.12X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.49 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.94 \\ 1.98 \\ 0 \end{bmatrix} = (3.94) \begin{bmatrix} 1 \\ 0.50 \\ 0 \end{bmatrix} = 3.94X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.50 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4X^{(6)}$$

$\because \quad X^{(6)} = X^{(5)}$

Hence the largest eigen value is 4 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

8. **Find the dominant eigen value and corresponding eigen vector of the matrix A by power method where:** $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$. **(April 2010)**

Sol. Let $X^{(0)} = (1,0,0)'$ be an approximate eigen vector.

$$AX^{(0)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = 5X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2\begin{bmatrix} 1 \\ 0 \\ 0.3846 \end{bmatrix} = 5.2X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.3846 \end{bmatrix} = \begin{bmatrix} 5.3846 \\ 0 \\ 2.9231 \end{bmatrix} = 5.3846\begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = 5.3846X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = \begin{bmatrix} 5.5429 \\ 0 \\ 3.7143 \end{bmatrix} = 5.5429\begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = 5.5429X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = \begin{bmatrix} 5.6701 \\ 0 \\ 4.3505 \end{bmatrix} = 5.6701\begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = 5.6701X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = \begin{bmatrix} 5.7642 \\ 0 \\ 4.8360 \end{bmatrix} = 5.7672\begin{bmatrix} 1 \\ 0 \\ 0.8385 \end{bmatrix} = 5.7672X^{(6)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8385 \end{bmatrix} = \begin{bmatrix} 5.8385 \\ 0 \\ 5.1927 \end{bmatrix} = 5.8385\begin{bmatrix} 1 \\ 0 \\ 0.8894 \end{bmatrix} = 5.8385X^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8894 \end{bmatrix} = \begin{bmatrix} 5.8894 \\ 0 \\ 5.4470 \end{bmatrix} = 5.8894\begin{bmatrix} 1 \\ 0 \\ 0.9249 \end{bmatrix} = 5.8894X^{(8)}$$

$$AX^{(8)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9249 \end{bmatrix} = \begin{bmatrix} 5.9249 \\ 0 \\ 5.6244 \end{bmatrix} = 5.9249\begin{bmatrix} 1 \\ 0 \\ 0.9493 \end{bmatrix} = 5.9249X^{(9)}$$

$$AX^{(9)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9493 \end{bmatrix} = \begin{bmatrix} 5.9493 \\ 0 \\ 5.7465 \end{bmatrix} = 5.9493\begin{bmatrix} 1 \\ 0 \\ 0.9659 \end{bmatrix} = 5.9493X^{(10)}$$

$$AX^{(10)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9659 \end{bmatrix} = \begin{bmatrix} 5.9659 \\ 0 \\ 5.8296 \end{bmatrix} = 5.9659\begin{bmatrix} 1 \\ 0 \\ 0.9771 \end{bmatrix} = 5.9659X^{(11)}$$

$$AX^{(11)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9771 \end{bmatrix} = \begin{bmatrix} 5.9771 \\ 0 \\ 5.8857 \end{bmatrix} = 5.9771\begin{bmatrix} 1 \\ 0 \\ 0.9847 \end{bmatrix} = 5.9771X^{(12)}$$

$$AX^{(12)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9847 \end{bmatrix} = \begin{bmatrix} 5.9847 \\ 0 \\ 5.9236 \end{bmatrix} = 5.987\begin{bmatrix} 1 \\ 0 \\ 0.9898 \end{bmatrix} = 5.9847X^{(13)}$$

$$AX^{(13)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9898 \end{bmatrix} = \begin{bmatrix} 5.9898 \\ 0 \\ 5.9489 \end{bmatrix} = 5.9898\begin{bmatrix} 1 \\ 0 \\ 0.9932 \end{bmatrix} = 5.9898X^{(14)}$$

$$AX^{(14)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9932 \end{bmatrix} = \begin{bmatrix} 5.9932 \\ 0 \\ 5.9659 \end{bmatrix} = 5.9932\begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = 5.9932X^{(15)}$$

$$AX^{(15)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = \begin{bmatrix} 5.9954 \\ 0 \\ 5.9772 \end{bmatrix} = 5.9954\begin{bmatrix} 1 \\ 0 \\ 0.9970 \end{bmatrix} = 5.9954X^{(16)}$$

$$AX^{(16)} = \begin{bmatrix} 5.9970 \\ 0 \\ 5.9848 \end{bmatrix} = 5.9970\begin{bmatrix} 1 \\ 0 \\ 0.9980 \end{bmatrix}$$

$\therefore \ \lambda_1 = 6$; Eigen vector $= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

9. Transform the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ to tridiagonal form using Householder's method.

(September 2009)

Sol. Given $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$s = \sqrt{a_{12}^2 + a_{13}^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$$

Taking $V = \begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}$

$$v_2^2 = \frac{1}{2}\left[1 + \frac{a_{12}}{s}\right] = \frac{1}{2}\left[1 + \frac{3}{\sqrt{10}}\right] = \frac{3+\sqrt{10}}{2\sqrt{10}}$$

$$2v_2v_3 = \frac{a_{13}}{s} \quad \Rightarrow \quad 2v_2v_3 = \frac{1}{\sqrt{10}}$$

$$\Rightarrow v_3^2 = \frac{1}{40v_2^2} = \frac{1}{2\sqrt{10}\left(3+\sqrt{10}\right)}$$

$$P = I - 2VV' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2v_2^2 & -2v_2v_3 \\ 0 & -2v_2v_3 & 1-2v_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Hence $A_1 = PAP$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ -\sqrt{10} & \frac{-8}{\sqrt{10}} & \frac{-7}{\sqrt{10}} \\ 0 & \frac{4}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -\sqrt{10} & 0 \\ -\sqrt{10} & \frac{31}{\sqrt{10}} & \frac{-13}{\sqrt{10}} \\ 0 & \frac{-13}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix}$$

10. Using Jacobi's method compute all the eigen values and the corresponding eigen vectors of the matrix: $\begin{bmatrix} 2 & \sqrt{2} & 4 \\ \sqrt{2} & 6 & \sqrt{2} \\ 4 & \sqrt{2} & 2 \end{bmatrix}$.

(April 2009)

Sol. The largest non-diagonal element is

$a_{13} = a_{31} = 4 > 0$

Since $a_{11} = a_{33} = 2$

$$\therefore \tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2 \times 4}{2-2} = \frac{8}{0} = \infty$$

$$\therefore 2\theta = \frac{\pi}{2} \qquad \Rightarrow \theta = \frac{\pi}{4}$$

The transformation matrix P_1 is $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore\ P_1^{-1} = P_1' = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

($\because$ P_1 is orthogonal matrix $\therefore$ $P_1P_1' = I$ so $P_1^{-1} = P_1'$)

The first transformation given $A_1 = P_1AP_1^{-1}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & \sqrt{2} & 4 \\ \sqrt{2} & 6 & \sqrt{2} \\ 4 & \sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

In A_1, the largest non-diagonal element is $a_{23} = a_{32} = 2 > 0$ and $a_{22} = a_{33} = 6$

$$\therefore\ \tan 2\theta = \frac{2a_{23}}{a_{22} - a_{33}} = \frac{2\times 2}{6-6} = \infty$$

$$\therefore\ 2\theta = \frac{\pi}{2} \qquad \Rightarrow \theta = \frac{\pi}{4}$$

Transformation matrix P_2 is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore\ P_2^{-1} = P_2' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The second transformation given

$A_2 = P_2A_1P_2^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \text{ which is a diagonal matrix}$$

$\therefore$ eigen values of given matrix are –2, 4, 8 and the corresponding eigen vectors are the columns of the product $P_1^{-1}P_2^{-1}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

7

NUMERICAL SOLUTION OF ODE

1. Apply Milne's method to find a solution of the differential equation. $\frac{dy}{dx}=1+xy^2$ at $x=0.8$, given that $y(0)=0, y(0.2)=0.2027, y(0.4)=0.4228$ and $y(0.6)=0.6841$. **(September 2013)**

Sol. Here $f(x,y)=1+xy^2$

Taking $h=0.2$ we have $x_0=0,\ x_1=0.2,\ x_2=0.4,\ x_3=0.6$

$y_0=0,\ y_1=0.2027,\ y_2=.4228,\ y_3=0.6841$

$f_0=1+x_0y_0^2=1+0\times 0=1$

$f_1=1+x_1y_1^2=1+(0.2)(0.2027)^2=1.0082$

$f_2=1+x_2y_2^2=1+(0.4)(0.4228)^2=1.0715$

$f_3=1+x_3y_3^2=1+(0.6)(0.6841)^2=1.2808$

To determine $y_4=y(0.8)$, we use Milne's predictor formula

$y_4=y_0+\frac{4h}{3}[2f_1-f_2+2f_3]$

$=0+\frac{4\times 0.2}{3}[2\times 1.0082-1.0715+2\times 1.2808]$

$=\frac{(0.8)(3.5065)}{3}$

$=0.9351$

$\therefore\ x_4=x_3+h=0.6+0.2=0.8$

$\Rightarrow f_4=1+x_4y_4^2=1+0.8\times(0.9351)^2=1.6995$

Using Milne's corrector formula, we get

$y_4^{(1)}=y_2+\frac{h}{3}[f_2+4f_3+f_4]$

$=0.4228+\frac{0.2}{3}[1.0715+4\times 1.2808+1.6995]$

$=0.94908$

$\therefore\ f_4^{(1)}=1+x_4\left(y_4^{(1)}\right)^2=1+0.8\times(0.94908)^2$

$=1.7206$

Again applying the corrector formula, we get

$y_4^{(2)}=y_2+\frac{h}{3}\left[f_2+4f_3+f_4^{(1)}\right]$

$=0.4228+\frac{0.2}{3}[1.0715+4\times 1.2808+1.7206]$

$=0.9505$

Hence $y(0.8)=0.9505$

2. If $y(x)$ satisfies the differential equation $y'=x-y^2$ with $y(0)=1$ use Taylor's series for $y(x)$ to find $y(0.1)$ correct to three decimal places. **(April 2013, September 2012)**

Sol. The given equation is $y'=x-y^2$ and $y(0)=1$

$\therefore\ x_0=0$ and $y_0=1$

$\therefore\ y_0'=x_0-y_0^2=0-1=-1$

$y_0''=1-2yy' \quad \therefore\ y_0''=1-2y_0y_0'=1+2=3$

$y'''=-2(y')^2-2yy'' \ \therefore y_0'''=-2(y_0')^2-2y_0y_0''$

$=-2\times 1-2\times 1\times 3=-8$

$y^{iv}=-4y'y''-2y'y''-yy'''=-6y'y''-2yy'''$

$\therefore\ y_0^{iv}=-6y_0'y_0''-2y_0y_0'''$

$=-6\times(-1)\times 3-2\times 1\times(-8)$

$=18+16=34$

$y_0^{v}=-6(y'')^2-6y'y'''-2y'y'''-2yy^{iv}$

$=-6(y'')^2-8y'y'''-2yy^{iv}$

$\therefore\ y_0^{v} = -6(y_0'')^2 - 8y_0'y_0''' - 2y_0y^{iv}_0$

$= -6(3)^2 - 8(-1)(-8) - 2(1)(34)$

$= -54 - 64 - 68 = -186$

The Taylor's series for $y(x)$ is given by

$$y(x) = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y_0^{v} + ...$$

$$= 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 + ...$$

To obtain the value of $y(0.1)$ correct to four decimal places, it is found that the terms upto x^4 should be considered.

$$\therefore\ y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 - (0.1)^3 - \frac{17}{12}(0.1)^4$$

$= 0.9138$

3. **Evaluate $y(0.2)$ from the differential equation $\frac{dy}{dx} = x^2 + y$, $y(0) = -1$, using Runge-Kutta method of 4[th] order.** **(April 2012)**

Sol. $f(x,y) = x^2 + y \quad x_0 = 0 \quad y_0 = -1 \quad h = 0.1$

Calculation of $y(0.1)$

$k_1 = hf(x_0, y_0) = (0.1)((0)^2 + (-1)) = -0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)((0.05)^2 + (-1.05)) = -0.10475$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)((0.05)^2 + (-1.0523)) = -0.10498$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)((.1)^2 + (-1.10498)) = -0.109498$$

$$y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= -1 + \frac{1}{6}(-0.1 - 0.2095 - 0.20996 - 0.109498)$$

$= -1.1048$

Calculation of $y(0.2)$ $\quad x_1 = 0.1 \quad y_1 = -1.1048 \quad h = 0.1$

$k_1 = hf(x_1, y_1) = (0.1)((0.1)^2 + (-1.1048)) = -0.10948$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)((0.15)^2 + (-1.15954)) = -0.113704$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)((0.15)^2 + (-1.161652)) = -0.113915$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)((0.2)^2 + (-1.218715)) = -0.117871$$

$$y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= -1.1048 + \frac{1}{6}(-0.10948 - 0.227408 - 0.22783 - 0.117871)$$

$= -1.2186$

4. **Use Runge-Kutta method of order 4 to estimate $y(1.2)$ of the equation: $\frac{dy}{dx} = x^2 + y^2, y(1) = 1.5$ with $h = 1.5$** **(September 2011)**

Sol. Here we have $f(x,y) = x^2 + y^2$, $x_0 = 1$, $y_0 = 1.5$, $h = 0.1$

$k_1 = hf(x_0, y_0) = (0.1)f(1,1.5) = 0.1\left[1^2 + (1.5)^2\right]$

$= 0.325$

$$k_2 = h\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f\left(1 + \frac{0.1}{2}, 1.5 + \frac{0.325}{2}\right)$$

$= (0.1)f(1.05,\ 1.6625)$

$= (0.1)\left[(1.05)^2 + (1.6625)^2\right]$

$= (0.1)(3.866) = 0.3866$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)f\left(1.05,\ 1.5 + \frac{0.3866}{2}\right)$$

$= (0.1)f(1.05,\ 1.6933)$

$= (0.1)\left[(1.05)^2 + (1.6933)^2\right]$

$= (0.1)(3.969) = 0.3969$

$k_4 = hf(x_0 + h,\ y_0 + k_3)$

$= (0.1)f(1 + 0.1,\ 1.5 + 0.3969)$

$=(0.1)f(1.1,\ 1.8969)$

$=(0.1)\left[(1.1)^2+(1.8969)^2\right]$

$=(0.1)(4.808)=0.4808$

$\therefore$ By Runge – Kutta fourth order method, we get

$y_1=y(1.1)=y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=1.5+\frac{1}{6}(0.325+2(0.3866)+2(0.3969)+0.4808)=1.8954$

To determine $y(1.2)$, we take $x_1=1.1$, $y_1=1.8954$ and $h=0.1$

$\therefore\ k_1=hf(x_1,y_1)=(0.1)f(1.1,\ 1.8954)$

$=(0.1)\left[(1.1)^2+(1.8954)^2\right]=0.4802$

$k_2=hf\left(x_1+\frac{h}{2},y_1+\frac{k_1}{2}\right)$

$=0.1f\left(1.1+\frac{0.1}{2},1.8954+\frac{0.4804}{2}\right)$

$=0.1f(1.15,\ 2.1355)$

$=0.1\left[(1.15)^2+(2.1355)^2\right]=0.5882$

$k_3=hf\left(x_1+\frac{h}{2},y_1+\frac{k_2}{2}\right)$

$=0.1f\left(1.15,\ 1.8954+\frac{0.5882}{2}\right)$

$=0.1f(1.15,\ 2.1895)$

$=(0.1)\left[(1.15)^2+(2.1895)^2\right]=0.6116$

$k_4=hf(x_1+h,y_1+k_3)=0.1f(1.1+0.1,\ 1.8954+0.6116)$

$=0.1f(1.2,\ 2.507)=0.7725$

$\therefore\ y_2=y(1.2)=y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=1.8954+\frac{1}{6}(0.4802+2\times0.5882+2\times0.6116+0.7725)$

$=2.5041$

Hence $y(1.2)=2.5041$

5. Given: $\frac{dy}{dx}-\sqrt{xy}=2, y(1)=1$, Find the value of $y(0.6)$ in steps of 0.1 using Euler method. (April 2011)

Sol. Given $\frac{dy}{dx}=2+\sqrt{xy}$, $y(1)=1$, $h=0.1$

$\therefore\ f(x,y)=2+\sqrt{xy}$

Applying Euler's formula

$y_{i+1}=y_i+hf(x_i,y_i)$

$y_1=y(1.1)=1+(0.1)\left(2+\sqrt{(1)(1)}\right)=1.3$

$y_2=y(1.2)=1.3+(0.1)\left(2+\sqrt{(1.1)(1.3)}\right)=1.6195$

$y_3=y(1.3)=1.6195+(0.1)\left(2+\sqrt{(1.2)(1.6195)}\right)=1.9589$

$y_4=y(1.4)=1.9589+(0.1)\left(2+\sqrt{(1.3)(1.9589)}\right)=2.3184$

$y_5=y(1.5)=2.3184+(0.1)\left(2+\sqrt{(1.4)(2.3184)}\right)=2.6985$

$y_6=y(1.6)=2.6985+(0.1)\left(2+\sqrt{(1.5)(2.6985)}\right)=3.0996$

Hence $y(1.6)=3.0996$

6. Given: $\frac{dy}{dx}=y-x$ where $y(0)=2$, find $y(0.1)$ and $y(0.2)$ correct to four decimal places. (April 2011)

Sol. To determine $y(0.1)$, we have $x_0=0$, $y_1=2$ and $h=0.1$

$k_1=hf(x_0,y_0)=h(y_0-x_0)=(0.1)(2-0)=0.2$

$k_2=hf\left(x_0+\frac{h}{2},y_0+\frac{k_1}{2}\right)$

$=(0.1)f\left(0+\frac{0.1}{2},2+\frac{0.2}{2}\right)$

$=(0.1)f(0.05,\ 2.1)$

$=(0.1)(2.1-0.05)$

$=(0.1)(2.05)$

$=0.205$

$k_3=hf\left(x_0+\frac{h}{2},y_0+\frac{k_2}{2}\right)$

$=(0.1)f\left(0+\frac{0.1}{2},2+\frac{0.205}{2}\right)$

$=(0.1)f(0.05,\ \ 2.1025)$

$=(0.1)(2.1025-0.05)$

$=(0.1)(2.0525)$

$=0.20525$

$k_4=hf(x_0+h,\ y_0+k_3)$

$=(0.1)f(0+0.1,\ \ 2+0.20525)$

$=(0.1)f(0.1,\ \ 2.20525)$

$=(0.1)(2.20525-0.1)$

$=(0.1)(2.10525)$

$=0.210525$

$=0.21053$ (approximately upto five decimal places)

$\therefore$ By Runge-Kutta fourth order formula

$y_1=y(0.1)=y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=2+\frac{1}{6}(0.2+2(0.205)+2(0.20525)+0.21053)$

$=2+0.2052$

$=2.2052$

To determiney (0.2), we use $x_1=0.1$, $y_1=2.2052$ and $h=0.1$

$k_1=hf(x_1,y_1)=(0.1)f(0.1,\ 2.2052)$

$=(0.1)(2.2052-0.1)=(0.1)(2.1052)=0.21052$

$k_2=hf\left(x_1+\frac{h}{2},\ y_1+\frac{k_1}{2}\right)$

$=(0.1)f\left(0.1+\frac{0.1}{2},\ 2.2052+\frac{0.21052}{2}\right)$

$=(0.1)f(0.15,\ 2.31046)$

$=(0.1)(2.31046-0.15)$

$=0.21605$

$k_3=hf\left(x_1+\frac{h}{2},\ y_1+\frac{k_2}{2}\right)$

$=(0.1)f\left(0.1+\frac{0.1}{2},\ 2.2052+\frac{0.21605}{2}\right)$

$=(0.1)f(0.15,\ 2.313225)$

$=(0.1)(2.313225-0.15)$

$=0.21632$

$k_4=hf(x_1+h,y_1+k_3)$

$=(0.1)f(0.1+0.1,\ 2.2052+0.21632)$

$=(0.1)f(0.2,\ 2.42152)$

$=(0.1)(2.42152-0.2)$

$=0.22215$

$\therefore\ y_2=y(0.2)=y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=2.2052+\frac{1}{6}[0.21052+2(0.21605)+2(0.21632)\ +0.22215]$

$=2.2052+0.2162$

$=2.4214$

7. **Use Runge-Kutta method of order 4 to compute $y(.2)$ if $y(0)=1$ and $h=.1$ and $10\frac{dy}{dx}=x^2+y^2$.** **(September 2010)**

Sol. The given differential equation can be written as

$\frac{dy}{dx}=\frac{x^2+y^2}{10},y(0)=1$

$\therefore\ f(x,y)=\frac{x^2+y^2}{10},\ x_0=0,\ \ y_0=1$

To determine y(0.1) we have $x_0=0,\ \ y_0=1$ and $h=0.1$

$k_1=hf(x_0,y_0)=h\left(\frac{x_0^{\ 2}+y_0^{\ 2}}{10}\right)=(0.1)\left(\frac{0+1}{10}\right)=0.01$

$k_2=hf\left(x_0+\frac{1}{2}h,\ \ y_0+\frac{k_1}{2}\right)$

$=h\left[\frac{\left(x_0+\frac{1}{2}h\right)^2+\left(y_0+\frac{k_1}{2}\right)^2}{10}\right]$

$$=(0.1)\left[\frac{\left(0+\frac{0.1}{2}\right)^2+\left(1+\frac{0.01}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.05)^2+(1.005)^2}{10}\right]$$

$$=(0.1)(0.10125)=0.010125$$

$$k_3=hf\left(x_0+\frac{1}{2}h,\ y_0+\frac{k_2}{2}\right)$$

$$=h\left[\frac{\left(x_0+\frac{1}{2}h\right)^2+\left(y_0+\frac{k_2}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.05)^2+\left(1+\frac{0.010125}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.05)^2+(1.00506)^2}{10}\right]$$

$$=(0.1)(0.1013)=0.01013$$

$$k_4=hf(x_0+h,\ y_0+k_3)$$

$$=h\left[\frac{(x_0+h)^2+(y_0+k_3)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0+0.1)^2+(1+0.01013)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.1)^2+(1.01013)^2}{10}\right]$$

$$=(0.1)\left[\frac{0.01+1.02036}{10}\right]$$

$$=0.0103$$

$$\therefore\ y_1=y(0.1)=y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$=1+\frac{1}{6}\left[0.01+2(0.010)+2(0.0101)+0.0103\right]$$

$$=1+\frac{1}{6}(0.06076)=1+0.01027=1.010127$$

$$=1.0101$$

To determine $y(0.2)$, we take $x_1=0.1,\ y_1=1.0101$ and $h=0.1$

$$k_1=hf(x_1,y_1)=h\left(\frac{x_1^2+y_1^2}{10}\right)$$

$$=(0.1)\left(\frac{(0.1)^2+(1.0101)^2}{10}\right)$$

$$=(0.1)(0.103)=0.0103$$

$$k_2=hf\left(x_1+\frac{1}{2}h,\ y_1+\frac{k_1}{2}\right)$$

$$=h\left[\frac{\left(x_1+\frac{1}{2}h\right)^2+\left(y_1+\frac{k_1}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{\left(0.1+\frac{0.1}{2}\right)^2+\left(1.0101+\frac{0.0103}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.15)^2+(1.0152)^2}{10}\right]$$

$$=0.01053$$

$$k_3=hf\left(x_1+\frac{1}{2}h,\ y_1+\frac{k_2}{2}\right)$$

$$=h\left[\frac{\left(x_1+\frac{1}{2}h\right)^2+\left(y_1+\frac{k_2}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.15)^2+\left(1.0101+\frac{0.01053}{2}\right)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.15)^2+(1.0154)^2}{10}\right]$$

$$=0.01054$$

$$k_4=hf(x_1+h,\ y_1+k_3)$$

$$=h\left[\frac{(x_1+h)^2+(y_1+k_3)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.1+0.1)^2+(1.0101+0.01054)^2}{10}\right]$$

$$=(0.1)\left[\frac{(0.2)^2+(1.0206)^2}{10}\right]=0.01082$$

$$\therefore\ y_2=y(0.2)=y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$=1.0101+\frac{1}{6}[0.0103+2(0.01053)\ +2(0.01054)+0.01082]$$

$$=1.0101+\frac{1}{6}(0.06326)$$

$$=1.0101+0.0105=1.0206$$

8. **Obtain Picard's second approximate solution of the differential equation:**

$\frac{dy}{dx}=\frac{x^2}{y^2+1};y(0)=0.$ **(April 2010)**

Sol. Given $f(x,y)=\frac{x^2}{y^2+1}, x_0=0,\ y_0=0$

$$y=y_0+\int_{x_0}^{x} f(x,y)\,dx=0+\int_0^x\frac{x^2}{y^2+1}dx$$

$$\text{or } y=\int_0^x\frac{x^2}{y^2+1}dx$$

First approximation. $y_1=\int_0^x\frac{x^2}{y_0^2+1}dx=\int_0^x x^2dx=\frac{x^3}{3}$

Second approximation. $y^2=\int_0^x\frac{x^2}{y_1^2+1}dx=\int_0^x\frac{x^2}{\frac{x^6}{9}+1}dx=\int_0^x\frac{x^2dx}{\left(\frac{x^3}{3}\right)^2+1}$

$$=\tan^{-1}\frac{x^3}{3}=\frac{x^3}{3}-\frac{x^9}{81}+\ldots\ldots$$

9. **Using Runge-Kutta method, find $y(0.2)$ given that $\frac{dy}{dx}=3x+\frac{1}{2}y, y(0)=1$, taking $h=0.1$.** **(September 2009)**

Sol. Here we have $f(x,y)=3x+\frac{y}{2}$

$$x_0=0,\ y_0=1,\ h=0.1$$

$$k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\left[3\times0+\frac{1}{2}\right]$$

$$=0.05$$

$$k_2=hf\left(x_0+\frac{h}{2},y_0+\frac{k_1}{2}\right)$$

$$=(0.1)f\left(0+\frac{0.1}{2},\ 1+\frac{0.05}{2}\right)$$

$$=(0.1)f(0.05,\ 1.025)$$

$$=(0.1)\left[3\times0.05+\frac{1.025}{2}\right]$$

$$=(0.1)(0.15+0.513)$$

$$=(0.1)(0.663)$$

$$=0.0663$$

$$k_3=hf\left(x_0+\frac{h}{2},\ y_0+\frac{k_2}{2}\right)$$

$$=(0.1)f\left(0+\frac{0.1}{2},\ 1+\frac{0.0663}{2}\right)$$

$$=(0.1)f(0.05,\ 1.0332)$$

$$=(0.1)\left[3\times0.05+\frac{1.0332}{2}\right]$$

$=(0.1)(0.15+0.5166)$

$=(0.1)(0.6666)$

$=0.0667$

$k_4 = hf(x_0+h,\ y_0+k_3)$

$=(0.1)f(0+0.1,\ 1+0.0667)$

$=(0.1)f(0.1,\ 1.0667)$

$=(0.1)\left[3\times0.1+\frac{1.0667}{2}\right]$

$=(0.1)(0.3+0.53335)$

$=(0.1)(0.833)=0.0833$

By Runge-Kutta fourth order, we get

$\therefore\ y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=1+\frac{1}{6}(0.05+2\times0.0663+2\times0.0667+0.0833)$

$=1+0.06655$

$=1.06655$

To obtain $y(0.2)$, we take $x_1=0.1$, $y=1.06655$ and $h=0.1$

$k_1 = hf(x_1,y_1)=(0.1)f(0.1,\ 1.06655)$

$=(0.1)\left[3\times0.1+\frac{1.06655}{2}\right]$

$=0.1\times0.833=0.0833$

$k_2 = hf\left(x_1+\frac{h}{2},\ y_1+\frac{k_2}{2}\right)$

$=(0.1)f\left(0.1+\frac{0.1}{2},\ 1.06655+\frac{0.0833}{2}\right)$

$=(0.1)f(0.15,\ 1.1082)$

$=(0.1)\left[3\times0.15+\frac{1.1082}{2}\right]$

$=(0.1)(1.0041)$

$=1.1004$

$k_3 = hf\left(x_1+\frac{h}{2},\ y_1+\frac{k_1}{2}\right)$

$=(0.1)f\left(0.1+\frac{0.1}{2},\ 1.06655+\frac{0.1004}{2}\right)$

$=(0.1)f(0.15,\ 1.1168)$

$=(0.1)\left[3\times0.15+\frac{1.1168}{2}\right]$

$=(0.1)(1.0084)=0.1008$

$k_4 = hf(x_1+h, y_1+k_3)$

$=(0.1)f(0.1+0.1,\ 1.06655+0.1008)$

$=(0.1)f(0.2,\ 1.1674)$

$=(0.1)\left[3\times0.2+\frac{1.1674}{2}\right]$

$=(0.1)(1.184)=0.1184$

$\therefore\ y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=1.06655+\frac{1}{6}(0.0833+2\times0.1004+2\times0.1008+0.1184)$

$=1.06655+0.1007=1.16725$

Hence $y(0.2)=1.16725$

10. Use Runge-Kutta method to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx}=x+y$ and y = 1 when x = 0. **(April 2009)**

Sol. Given $\frac{dy}{dx}=x+y,\qquad y(0)=1$

Calculation of $y(0.1)$:

$k_1 = hf(x_0,y_0)=(0.1)(0+1)=0.1$

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)=(0.1)(0.05+1.05)=0.11$

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)=(0.1)(0.05+1.055)=0.1105$

$k_4 = hf(x_0+h, y_0+k_3)=(0.1)(0.1+1.1105)=0.12105$

Using Runge-Kutta 4th order method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = 1 + \frac{1}{6}\left(0.1 + 2(0.11 + 0.1105) + 0.12105\right) = 1.11034$$

Calculation of $y(0.2)$:

$$k_1 = hf(x_1, y_1) = (0.1)(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)(0.15 + 1.170857) = 0.132085$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)(0.15 + 1.176382) = 0.132638$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)(0.2 + 1.242978) = 0.144298$$

Using Runge-Kutta 4th order method

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_2 = 1.11034 + \frac{1}{6}\left(0.121034 + 2(0.132085 + 0.132638) + 0.144298\right)$$

$= 1.2428$

Hence $y(0.2) = 1.2428$

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